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http://www3.math.tu-berlin.de/geometrie/Lehre/SS12/ComplexAnalysis SS 12

Complex Analysis - Exercise Sheet 10

- More residue calculations -

due 27-28.06.2012

Tutorial

Exercise 1

Given a meromorphic function $f(z)$ and a holomorphic function $h(z)$. Calculate the residues of $h(z)\frac{f'(z)}{f(z)}$.

Exercise 2

Find all functions f having all the following three properties:

1. f has a double pole at $z = 0$ with residue 2,
2. f has a simple pole at $z = 1$ with residue 2,
3. f is bounded in a neighborhood of ∞.

Exercise 3

Verify the following integrals:

1. $\displaystyle\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}dx = \frac{\pi}{\sqrt{3}}$
2. For $a > 1$, $\displaystyle\int_0^\pi \frac{d\theta}{a+\cos(\theta)} = \frac{\pi}{\sqrt{a^2-1}}$.
3. For $a \geq 0$ and $b > 0$, $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+b^2}dx = \frac{\pi e^{-ab}}{2b}$

Homework

Exercise 1 **12 points**

Verify the following integrals:

1. For $0 < p < 1$, $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2p\cos(\theta) + p^2} = \frac{2\pi}{1 - p^2}$
2. For $0 < p < 1$, $\displaystyle\int_0^{\infty} \frac{x^{p-1}}{1 + x} dx = \frac{\pi}{\sin p\pi}$
3. $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \frac{\pi}{12}.$

Exercise 2 **8 points**

Let ω_1 and $\omega_2 \in \mathbb{C}$ be real linearly independent and let f be a meromorphic, doubly-periodic function with periods ω_1 and ω_2, that is,

$$f(z + \omega_i) = f(z).$$

Define the *fundamental domain* $F \subset \mathbb{C}$:

$$F := \{\lambda_1\omega_1 + \lambda_2\omega_2 \mid 0 \leq \lambda_i < 1\}$$

Let z_{0j} and $z_{\infty j}$ be the zeroes and poles, resp., of f, in the fundamental domain F, with multiplicities n_j and m_j, resp. Assume no zero or pole lies on the boundary of F.

1. Show that f has the same number of poles and zeroes counted with multiplicity on F.
2. Show there exists integers k_1 and k_2 such that

$$\sum_j n_j z_{0j} - \sum_j m_j z_{\infty j} = k_1\omega_1 + k_2\omega_2$$

   That is, the sum of the *locations* of the zeros and poles, counted with multiplicity, lies on the lattice generated by the periods ω_1 and ω_2. Hint: apply Tutorial Exercise 1 with appropriate choice of $h(z)$.