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Inventiones mathematicae

Hopf tori in S^3

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1. Introduction

Let $\pi: S^3 \to S^2$ be the Hopf fibration. Then the inverse image of any closed curve on S^2 will be an immersed torus in S^3 which will be called a *Hopf torus*. Using Hopf tori we prove

Theorem 1. *Every compact Riemann surface of genus one can be conformally embedded in the unit sphere $S^3 \subset \mathbb{R}^4$ as a flat torus. The embedding can be chosen as the intersection of S^3 with a quartic hypersurface in $\mathbb{R}^4$.*

By stereographic projection of S^3 onto $\mathbb{R}^3$ we obtain the following

Corollary. *Every compact Riemann surface of genus one can be conformally embedded in $\mathbb{R}^3$ as an algebraic surface of degree eight.*

Garsia [2] had shown that every compact Riemann surface (of any genus) can be conformally embedded in $\mathbb{R}^3$ as an algebraic surface, but his method of proof was not constructive and he therefore did not obtain bounds for the degree of this surface. As another application of Hopf tori we construct new examples of compact embedded Willmore surfaces. A surface in $\mathbb{R}^3$ is called a *Willmore surface* if it is an extremal surface for the variational functional $\int H^2 dA$ (H the mean curvature) [1]. The only examples of compact embedded Willmore surfaces known so far are the stereographic projections of compact embedded minimal surfaces in S^3 [9]. Using results of Langer and Singer [3] on elastic curves on S^2 we will exhibit an infinite series of compact embedded Willmore surfaces that do not stem from minimal surfaces in S^3.

2. Hopf tori

The usual way to describe the Hopf map $\pi: S^3 \to S^2$ is to restrict the canonical projection of $\mathbb{C}^2 - \{0\}$ onto $\mathbb{C}P^1 = S^2$ to the unit sphere S^3 in $\mathbb{C}^2$. For our purpose we need a more explicit description.

We identify S^3 with the set of unit quaternions $\{q \in \mathbb{H},\ q\bar{q}=1\}$ and S^2 with the unit sphere in the subspace of $\mathbb{H}$ spanned by $1, j$ and k. Let $q \mapsto \tilde{q}$ denote the antiautomorphism of $\mathbb{H}$ that fixes $1, j$ and k but sends i to $-i$. Define $\pi: S^3 \to \mathbb{H}$ by

$$\pi(q) = \tilde{q}q. \tag{1}$$

Then π has the following properties:

a) $\pi(S^3) = S^2$.

b) $\pi(e^{i\varphi} q) = \pi(q)$ for all $q \in S^3$, $\varphi \in \mathbb{R}$.

c) The group S^3 acts isometrically on S^3 by right multiplication and on S^2 via

$$q \to \tilde{r}qr, \quad r \in S^3. \tag{2}$$

π intertwines these two actions, i.e. for all $q, r \in S^3$ we have $\pi(qr) = \tilde{r}\pi(q)r$.

Let $\mathfrak{p}: [a, b] \to S^2$ be an immersed curve. Choose $\mathfrak{y}: [a, b] \to S^3$ such that $\pi \circ \mathfrak{y} = \mathfrak{p}$. Then with the notation $S^1 := \mathbb{R}/2\pi\mathbb{Z}$ we define an immersion $\mathfrak{x}$ of the cylinder $[a, b] \times S^1$ into S^3 by

$$\mathfrak{x}(t, \varphi) = e^{i\varphi}\, \mathfrak{y}(t). \tag{3}$$

$\mathfrak{x}$ is called the *Hopf cylinder* corresponding to the curve $\mathfrak{p}$. We always assume that the curve $\mathfrak{y}$ is parametrized by arclength and cuts the fibres of π orthogonally, that means $\mathfrak{y}'(t)$ has unit norm for all t and is orthogonal to

$$\mathfrak{x}_\varphi(t, \varphi) = ie^{i\varphi}\, \mathfrak{y}(t). \tag{4}$$

Lower indices always will denote partial derivatives. Since in addition $\mathfrak{y}'(t)$ is orthogonal to $\mathfrak{y}(t)$ there is a function $u: [a, b] \to \operatorname{span}(j, k)$ such that $|u| = 1$ and

$$\mathfrak{y}' = u\mathfrak{y}. \tag{5}$$

Concerning the curve $\mathfrak{p} = \pi \circ \mathfrak{y}$ on S^2 we have

$$\begin{aligned} \mathfrak{p} &= \tilde{\mathfrak{y}}\mathfrak{y} \\ \mathfrak{p}' &= 2\tilde{\mathfrak{y}}u\mathfrak{y} \\ |\mathfrak{p}'| &= 2. \end{aligned} \tag{6}$$

The two partial derivatives

$$\begin{aligned} \mathfrak{x}_\varphi &= e^{i\varphi} i\mathfrak{y} \\ \mathfrak{x}_s &= e^{i\varphi} u\mathfrak{y} \end{aligned} \tag{7}$$

are orthogonal and have unit length, hence (6) implies

Lemma 1. *Let $\mathfrak{p}$ be a curvilinear arc on S^2 of length L. Then the corresponding Hopf cylinder is isometric to $[0, L/2] \times S^1$.*

If $\mathfrak{p}$ is a closed curve (i.e. $\mathfrak{p}(t + L/2) = \mathfrak{p}(t)$ for all t) then equation (4) defines a covering of the (t, φ)-plane onto an immersed torus in S^3. This torus will be called the *Hopf torus* corresponding to $\mathfrak{p}$. The isometry type of this torus depends not only on the length of $\mathfrak{p}$ but also on the "area enclosed by $\mathfrak{p}$": If $\mathfrak{p}: [a, b] \to S^2$ is any curve with $\mathfrak{p}(b) = \mathfrak{p}(a)$ then we define the "oriented area enclosed by $\mathfrak{p}$" as

$$A = \int_c dV \tag{8}$$

where dV is the canonical volume form on S^2 and c is an arbitrary 2-chain on S^2 such that $\partial c = \mathfrak{p}$ and $\int_c dV \in [-2\pi, 2\pi)$. A is well defined because $H_2(S^2) \simeq \mathbb{Z}$ and $\mathrm{vol}(S^2) = 4\pi$.

Proposition 1. *Let $\mathfrak{p}$ be a closed curve on S^0 of length L enclosing an oriented area A. Then the corresponding Hopf torus M is isometric to $\mathbb{R}^2/\Gamma$, where the lattice Γ is generated by the vectors $(2\pi, 0)$ and $(A/2, L/2)$.*

Proof. Let $\mathfrak{y}$ and $\mathfrak{x}$ be defined as in (3). Then $\mathfrak{x}$ can be considered as a Riemannian covering of the (t, φ)-plane onto M. The translation in the direction $(2\pi, 0)$ generates a group of deck transformations for this covering. The lines in Fig. 1 parallel to the φ-axis are mapped by $\mathfrak{x}$ onto fibres of π.

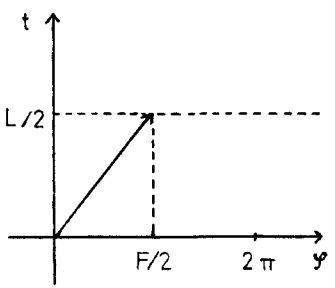

Fig. 1

The t-axis is mapped by $\mathfrak{x}$ onto the curve $\mathfrak{y}$. We have $\pi(\mathfrak{y}(L/2)) = \pi(\mathfrak{y}(0))$ and hence

$$\mathfrak{y}(L/2) = e^{i\delta}\,\mathfrak{y}(0) \tag{9}$$

for some $\delta \in [-\pi, \pi)$. It is clear that the whole group of deck transformations for the covering $\mathfrak{x}$ is generated by the translations $(2\pi, 0)$ and $(\delta, L/2)$. To prove the proposition we have to show

$$\delta = A/2. \tag{10}$$

Note first that the mapping $\pi: S^3 \to S^2$ can also be considered as a principal fibre bundle over S^2 with structure group S^1 (a "circle bundle"). We define a connection on this bundle by assigning to each $x \in S^3$ the subspace of $T_x S^3$ orthogonal to the fibre of π through x. Then the curve $\mathfrak{y}$ is a "horizontal lift" for $\mathfrak{p}$. Let $\Omega \in \Lambda^2(S^2)$ be the curvature 2-form of the above connection. The Euler number of the circle bundle π is 1, hence we have

$$\int_{S^2} \Omega = 2\pi. \tag{11}$$

For reasons of symmetry (see (c) on p. 2) Ω must be a multiple of the volume form dV, thus by (11)

$$\Omega = 1/2\, dV. \tag{12}$$

Now it is well known that the curvature form of a circle bundle measures the non-closedness of horizontal lifts of closed curves. In our situation this means

$$\delta = \int_c \Omega \tag{13}$$

where c is any 2-chain on S^2 such that $\partial c = \mathfrak{p}$ and $\int_c \Omega \in [-\pi, \pi)$. (The proof of the theorem on p. 191 of [8] can be easily adapted to yield this result.) The proposition now follows from (8), (12) and (13). □

3. Algebraic flat tori in S^3

Let γ be an embedded closed curve on S^2 of length L and oriented area A. Changing the orientation on γ changes the sign of A. So we can assume $0 \leqq A \leqq 2\pi$. The only further restriction on (A, L) is given by the isoperimetric inequality on the sphere [7]:

$$L^2 - 4\pi A - A^2 \geqq 0. \tag{14}$$

Equality in (14) is attained only for circles on S^2. In terms of $(A/2, L/2)$ we can write (14) as

$$(A/2 - \pi)^2 + (L/2)^2 \geqq \pi^2. \tag{15}$$

Thus for each point $(A/2, L/2)$ in the shaded region of Fig. 2 there is an embedded closed curve on S^2 with length L and area A.

It is well known that every compact Riemann surface of genus one is conformally equivalent to $\mathbb{R}^2/\Gamma$ where Γ is the lattice in $\mathbb{R}^2$ generated by $(2\pi, 0)$ and another vector whose endpoint lies in the doubly shaded region of Fig. 2. Thus we have already proved the first of the two assertions in Theorem 1.

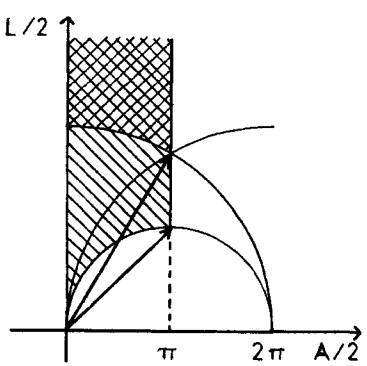

Fig. 2

Examples. 1) The point $(A/2, L/2) = (\pi, \pi)$ corresponds to a square lattice in $\mathbb{R}^2$. But $A = L = 2\pi$ is attained for a great circle on S^2, hence the inverse image $\pi^{-1}(\gamma)$ of a great circle γ on S^2 under the Hopf map π is isometric to a square torus. In fact one can verify that $\pi^{-1}(\gamma)$ is a "Clifford torus". Under a suitable stereographic projection $\pi^{-1}(\gamma)$ is mapped onto a special standard torus. In Fig. 3 also the images of the Hopf fibres are indicated.

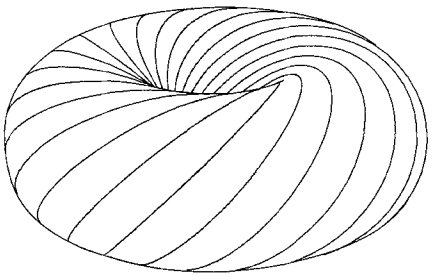

Fig. 3

2) The point $(A/2, L/2)=(\pi, \sqrt{3}\pi)$ corresponds to a hexagonal lattice Γ in $\mathbb{R}^2$. To obtain a torus which is conformally equivalent to the hexagonal torus $\mathbb{R}^2/\Gamma$ we only have to chose any curve γ on S^2 with $A=2\pi$, $L=\sqrt{3}\,2\pi$ and then consider $\pi^{-1}(\gamma)$. One possible choice for γ is shown in Fig. 4a). Here γ is the intersection of S^2 with a cubic cone. Since π is given by quadratic polynomials (see (1)) the corresponding Hopf torus $\pi^{-1}(\gamma)$ is the intersection of S^3 with a hypersurface in $\mathbb{R}^4$ of degree 6. Thus $\pi^{-1}(\gamma)$ is algebraic of degree 12. The stereographic projection of S^3-{northpole} onto $\mathbb{R}^3$ can be considered as the restriction of a *linear* projection $\mathbb{R}^4$-{northpole} $\to \mathbb{R}^3$. Since linear projections preserve degree, also the stereographic projection of $\pi^{-1}(\gamma)$ has algebraic degree 12 (Fig. 4b).

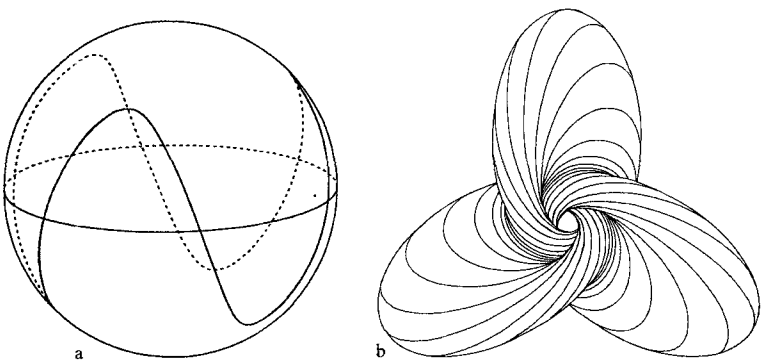

Fig. 4

We now complete the proof of Theorem 1 by showing that one can achieve any conformal structure on $\pi^{-1}(\gamma)$ by choosing γ to be a suitable curve on S^2 of degree less than or equal to four.

If the conformal structure in question corresponds to a rectangular lattice then we can chose γ to be a suitable circle on S^2. In this case we have equality in (14) and (15).

If the conformal structure is not rectangular then we chose a corresponding point (A, L) in the region $U \subset \mathbb{R}^2$ defined by the inequalities

$$\begin{gathered} 0<A/2<2\pi \\ \sqrt{\pi^2-(A/2-\pi)^2}<L/2<2\sqrt{\pi^2-(A/2-\pi)^2}. \end{gathered} \tag{16}$$

This choice is possible because U contains nearly a whole fundamental region of the modular group (see Fig. 2). Only the points on the circle $L/2 = \sqrt{\pi^2 - (A/2-\pi)^2}$ are missing, which correspond however to rectangular lattices.

Let D_1 be a circular disk on S^2 with area A and denote the length of its boundary circle by L_1. By (16) we have $L_1 < L < 2L_1$, hence $0 < L - L_1 < L_1$. Let D_2 be another circular disk on S^2 such that the length of ∂D_2 is equal to $L_2 = L - L_1$ and

$$\text{area}(D_1 \cap D_2) = 1/2 \text{ area } (D_2). \tag{17}$$

It is easy to see that such D_2 always exists and that the configuration consisting of the two disks D_1, D_2 is uniquely defined by the above conditions up to congruent motions of S^2.

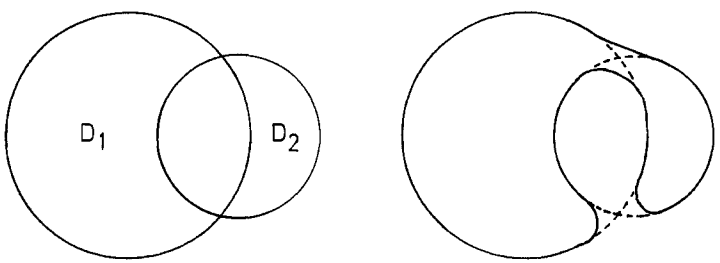

Fig. 5

The union of the two boundary circles ∂D_1 and ∂D_2 can be considered as a reducible quartic curve. As indicated in Fig. 5 we can perturb this reducible quartic slightly so as to obtain a nonsingular and connected quartic γ whose length $\tilde{L}$ and area $\tilde{A}$ are approximately given by L and A. Obviously we can parametrize this perturbation by a parameter ε with $0 \leqq \varepsilon < 1$ such that $\tilde{L}$ and $\tilde{A}$ depend contineously on L, A and ε. The function

$$f = (\tilde{A}, \tilde{L}): \quad U \times [0,1) \to \mathbb{R}^2 \tag{18}$$

obtained in this way satisfies the condition $f(x, 0) = x$ for all $x \in U$. Now the following lemma provides us with a non-singular connected quartic on S^2 with prescribed area and length $(A, L) \in U$:

Lemma 2. *Let $U \subset \mathbb{R}^2$ be open, $f: U \times [0,1) \to \mathbb{R}^2$ a contineous mapping such that $f(x, 0) = x$ for all $x \in U$. Then for each $x \in U$ there is a pair $(y, \varepsilon) \in U \times (0,1)$ such that $f(y, \varepsilon) = x$.*

The proof of Lemma 2 is left to the reader.

3. Willmore tori

We now want to determine the mean curvature of a Hopf torus. By (5) we have

$$\begin{aligned} \mathfrak{x}_t(t, \varphi) &= e^{i\varphi} u(t)\, \mathfrak{y}(t) \\ &= u(t) e^{-i\varphi} \mathfrak{y}(t). \end{aligned} \tag{19}$$

Using (4) and (19) it can be verified that a unit normal vector field for the immersion $\mathfrak{x}$ is given by

$$\mathfrak{n}(t, \varphi) = iu(t)\, e^{-i\varphi}\, \mathfrak{y}(t). \tag{20}$$

Taking derivatives of (20) we obtain

$$\begin{aligned} \mathfrak{n}_t &= -2\kappa \mathfrak{x}_t - \mathfrak{x}_\varphi \\ \mathfrak{n}_\varphi &= -\mathfrak{x}_t. \end{aligned} \tag{21}$$

Here we have defined κ by the equation

$$u' = 2i\kappa u. \tag{22}$$

$t \mapsto \kappa(t)$ is the curvature function of the spherical curve $\mathfrak{p} = \bar{\mathfrak{y}}\mathfrak{y}$. This can be seen from the identities

$$\begin{aligned} (\bar{\mathfrak{y}}\mathfrak{y})' &= 2\bar{\mathfrak{y}} u \mathfrak{y} \\ (\bar{\mathfrak{y}}\mathfrak{y})'' &= 2\bar{\mathfrak{y}}(u' - 2)\mathfrak{y}. \end{aligned} \tag{23}$$

(Recall that for $\mathfrak{z} \in S^3$ the map sending $q \in S^2$ to $\bar{\mathfrak{z}} q \mathfrak{z}$ is an isometry of S^2).

By (21) $\kappa(t)$ is also the mean curvature of the surface $\mathfrak{x}$ at the point $\mathfrak{x}(t, \varphi)$. Therefore the Willmore functional $\mathfrak{W}(M)$[1, 9] is given by

$$\begin{aligned} \mathfrak{W}(M) &= \int_F 1 + \kappa^2(t)\, d\varphi\, dt \\ &= \pi \int_0^L 1 + \kappa^2(s)\, ds. \end{aligned}$$

Here F is a fundamental region for the covering $\mathfrak{x}: \mathbb{R}^2 \to M$ and $ds = 2dt$ denotes arclength on the spherical curve $\mathfrak{p}$. The principle of symmetric criticality [5] is applicable here, hence $\mathfrak{x}$ is an extremal surface for the functional $\mathfrak{W}$ if and only if $\mathfrak{p}$ is an extremal curve for $\int 1 + \kappa^2\, ds$.

Langer and Singer [3] have shown that there are infinitely many simple closed curves on S^2 that are critical points for $\oint 1 + \kappa^2\, ds$. Therefore there are infinitely many embedded Hopf tori that are critical points for $\mathfrak{W}$. The stereographic projections of these tori are then embedded Willmore tori in $\mathbb{R}^3$. Figure 6 shows such a Willmore torus and the corresponding curve on S^2.

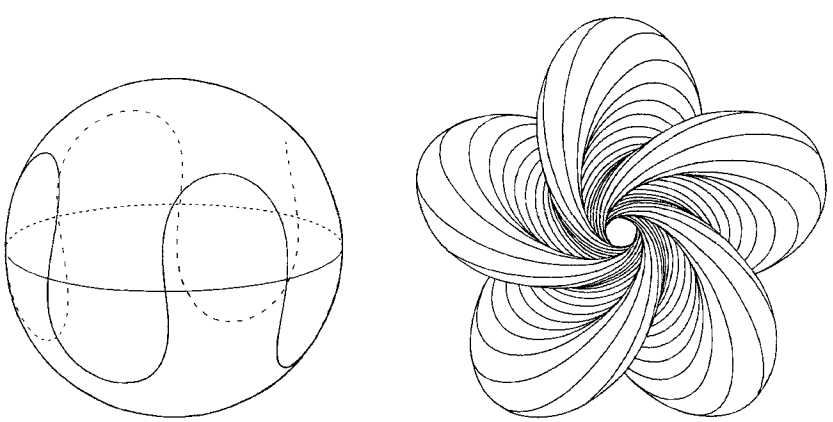

Fig. 6

We now want to show that with one exception the above examples of Willmore tori in $\mathbb{R}^3$ cannot be obtained by stereographic projection from minimal surfaces in S^3. Clearly this follows from

Proposition 2. *Let $M \subset S^3$ be a Hopf torus that is a critical point of $\mathfrak{W}$, $\alpha: S^3 \to S^3$ a conformal transformation such that $\alpha(M)$ is a minimal surface in S^3. Then M is a Clifford torus.*

Proof. The directions of curvature on a minimal surface N in S^3 are given by the zero directions of the real part of a holomorphic quadratic differential on N [4]. This property is invariant under conformal transformations, hence under the hypotheses of the proposition the same must hold for M. Using the complex coordinate $z = t + i\varphi$ on M (t and φ defined as in (3)) we can write the mentioned quadratic differential on M as $a dz^2$ for some constant $a \in \mathbb{C}$. This means in particular that on M the lines of curvature cut the Hopf fibres under a constant angle. Then it can be seen from (21) that κ must be constant, that means the curve γ on S^2 corresponding to M is a circle. Since by our assumptions on M this circle is a critical point for the functional $\oint 1 + \kappa^2 \, ds$ it must be a great circle. □

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