TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

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Complex Analysis I - SS 14

http://www3.math.tu-berlin.de/geometrie/Lehre/SS14/ComplexAnalysis/

8. Homework

Exercise 1 (4 Points)

Let $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ and $f : \mathbb{D} \to U \subset \mathbb{C}$ conformal (i.e., f is bijective and f and f^{-1} are holomorphic). Show that

$$|f'(0)| \geq \text{dist}(f(0), \partial U).$$

Remark: $\text{dist}(f(p), \partial U) := \sup\{R > 0 | K_R(p) \subset U\}$, where $K_R(p)$ is a disk of radius R around $p \in U$

Exercise 2 (8 Points)

Define the *hyperbolic length* of a continuously differentiable curve $\gamma : [0, 1] \to \mathbb{D}$ by

$$L_H(\gamma) := \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2}.$$

For two points z_0, z_1 in $\mathbb{D}$ we define the *hyperbolic distance* by

$$d_H(z_0, z_1) := \inf_{\substack{\gamma:[0,1]\to\mathbb{D} \\ \gamma(0)=z_0, \gamma(1)=z_1}} L_H(\gamma).$$

a) Show that the hyperbolic distance d_H is invariant under Möbius transformations $f : \mathbb{D} \to \mathbb{D}$.

b) Determine $d_H(0, r)$ for $r \in]0, 1[$. For which curve do you attain the infimum of the above definition (i.e., a 'shortest hyperbolic connection')?

c) For which other points z_0, z_1 are you in principal able to compute the hyperbolic distance $d_H(z_0, z_1)$? How does the shortest (hyperbolic) connection look in that case? Thinking about these arcs as 'hyperbolic line segments', identify the 'hyperbolic straight lines'.

d) Explain why d_H yields a metric on $\mathbb{D}$.

e) Show that every holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$ is a contraction in the hyperbolic metric d_H, i.e. $d_H(f(z_0), f(z_1)) \leq d_H(z_0, z_1)$ for all $z_0, z_1 \in \mathbb{D}$.
Hint: Use exercise 3 from sheet 7.

Homework due in the tutorial 18./19.6.14 or in the mailbox in front of MA 871 by 19.6.14 2pm latest.