TECHNISCHE UNIVERSITÄT BERLIN
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Complex Analysis - SS 2015

http://www3.math.tu-berlin.de/geometrie/Lehre/SS15/ComplexAnalysis/

Exercise Sheet 1

Exercise 1: Scale rotations and complex numbers. (6 pts)

Consider the set of scale rotations in $\mathbb{R}^2$

$$\mathcal{C} := \mathbb{R} \cdot \mathrm{SO}(2) = \left\{ \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \mid a, b \in \mathbb{R} \right\}.$$

Together with addition and matrix multiplication $\mathcal{C}$ becomes a field (note that each two elements of $\mathcal{C}$ commute).

(i) Show that any linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ belongs to $\mathcal{C} \setminus \{0\}$ if and only if $\det A > 0$ and there is some $r > 0$ such that

$$\langle Ax, Ay \rangle = r \langle x, y \rangle \qquad \text{for all } x, y \in \mathbb{R}^2.$$

(ii) Show that the map

$$\Phi : \mathbb{C} \to \mathcal{C}, \quad a + ib \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$$

is an isomorphism of fields.

Exercise 2: Complex numbers and plane geometry. (7 pts)

(i) For which $a, b, c, d \in \mathbb{C}$ does the complex equation

$$az\bar{z} + bz + c\bar{z} + d = 0$$

represent a line, for which a circle?

(ii) Write the equation of an ellipse centered at the origin,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{for some } a, b > 0,$$

in complex form.

(iii) Using complex numbers, prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

Exercise 3: Cauchy-Riemann equations and differentiability. (7 pts)

(i) Show that the functions $f, g : \mathbb{C} \to \mathbb{C}$ given by

$$f(x+iy) := -4xy + 2i(x^2 - y^2 + 3)$$
$$g(x+iy) := x^2 + y^2 + 2ixy$$

are real differentiable, and check which of them satisfies the Cauchy-Riemann equations, i.e. is holomorphic.
Express both functions in terms of z and $\bar{z}$ only.

(ii) Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(x+iy) := \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}.$$

Show that $\lim_{z\to 0} \frac{f(z)-f(0)}{z}$ is equal to 0 if $z \to 0$ along any straight line, and it is equal to $\frac{1}{2}$ if $z \to 0$ along the curve $x = y^2$.
Conclude that f satisfies the Cauchy-Riemann equations at $z = 0$, but it is not differentiable at $z = 0$ (neither complex nor real).

Due Tuesday, May 05, before the lecture.