TECHNISCHE UNIVERSITÄT BERLIN
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Complex Analysis - SS 2015

http://www3.math.tu-berlin.de/geometrie/Lehre/SS15/ComplexAnalysis/

Exercise Sheet 7

Exercise 1: Power series expansion. (5 pts)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic (i.e. f an entire function). Let $A, R > 0$ and $m \in \mathbb{N}$ be constants such that

$$|f(z)| \leq A\,|z|^m \quad \text{for all } z \in \mathbb{C} \text{ with } |z| > R.$$

Show that f is a polynomial of degree at most m.

Exercise 2: Maximum Principle. (5 pts)

Let $G \subset \mathbb{C}$ be a domain and $K \subset G$ a compact subset with non-empty interior $\overset{\circ}{K}$. Show that every non-constant holomorphic function on G with constant absolute value on the boundary $\partial K = K \setminus \overset{\circ}{K}$ has a zero in K.

Exercise 3: Reflection Principle. (5 pts)

Show: If f is an entire function such that $f(\mathbb{R}) \subset \mathbb{R}$ and $f(i\mathbb{R}) \subset i\mathbb{R}$, then f is an odd function, i.e. $f(-z) = -f(z)$.

Exercise 4: Zeros of holomorphic functions. (5 pts)

Let G be a domain and $f : G \to \mathbb{C}$ a holomorphic map. Let $M := \{z \in G \mid f(z) \subset \mathbb{R}\}$ and $z_0 \in M$. Prove the following statements:

(i) If $f'(z_0) \neq 0$, then there exists a neighborhood V of z_0 in G such that $M \cap V$ is the image of a curve $\gamma : (0, 1) \to G$.

(ii) If $f'(z_0) = 0$, i.e. $n := \operatorname{Ord}(f - f(z_0), z_0) = \min\left\{k \in \mathbb{N} \mid f^{(k)}(z_0) \neq 0\right\} \geq 2$, then there exists a neighborhood V of z_0 in G such that $M \cap V$ is the union of n curves. Furthermore, the intersection angles of these curves at z_0 are multiples of $\frac{2\pi}{n}$.

Bonus: (3 pts)

Calculate and draw the set M for the following maps: exp, $z \mapsto z^3$, sin.

Due Friday, June 05, before the lecture.