- <u>parametric surfaces</u>

graphs ruled surfaces. surfaces of revolution.

$(x, y, f(x,y))$ $\beta(t) + u\delta(t)$ $(r(t)\cos\theta,\ r(t)\sin\theta,\ z(t))$.

- <u>immersion</u> regular surface $X_u \times X_v \neq 0$.

- <u>tangent space</u> (plane) $T_pM := \mathrm{Span}\langle X_u, X_v\rangle$

$= \{$velocity vectors to curves in M through $p\}$

<u>normal line</u> $N_pM := \mathrm{Span}\langle X_u \times X_v\rangle$.

<u>unit normal vector</u> $\nu := \dfrac{X_u \times X_v}{|X_u \times X_v|}$ $\rightsquigarrow$ orientation. (locally).

- <u>first fundamental form</u> - inner product on T_pM in heherited from $T_p\mathbb{R}^3 = \mathbb{R}^3$

- associated quadratic form.

$$g_p = I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle X_u, X_u\rangle & \langle X_u, X_v\rangle \\ \langle X_v, X_u\rangle & \langle X_v, X_v\rangle \end{pmatrix}$$

$\det g_p = EG - F^2 = |X_u \times X_v|^2 \geq 0$. (imersion $\Leftrightarrow$ ~~deg~~ $\det g > 0$).

- <u>length</u> of β $\mathrm{len}\,\beta = \int \sqrt{g(\dot\beta)}\,dt = \int |\dot\beta|\,dt$.

<u>area</u> of M (surface patch) area $= \int_U |X_u \times X_v|\,du\,dv = \int_U \sqrt{\det g}\,du\,dv$.

- <u>smooth maps</u> $f: \mathbb{R}^m \supseteq A \longrightarrow \mathbb{R}^n$ <u>smooth</u> if $\exists\, U$ open. $U \supseteq A$.

and extension) $\bar f: U \longrightarrow \mathbb{R}^n$. s.t. $\bar f|_U = f$ and $\bar f$ smooth.

$f: M \longrightarrow \mathbb{R}^n$ smooth $\Leftrightarrow$ $f \circ X$ smooth

<u>smoothness in coordinates</u>.

$f: \mathbb{R}^n \longrightarrow M$ smooth $\Leftrightarrow$ $X^{-1} \circ f$ smooth

$f: M \longrightarrow N$ smooth $\Leftrightarrow$ $Y^{-1} \circ f \circ X$ smooth

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, "X"] \arrow[r] & V \arrow[u, "Y"'] \end{tikzcd}$$

- <u>differential</u>

$f: M \longrightarrow N$. $D_pf: T_pM \longrightarrow T_{f(p)}N$.

- <u>Gauss map</u> $\nu : M \longrightarrow S^2$

<u>shape operator</u> / Weingarten operator $S_p = -D_p\nu : T_pM \longrightarrow T_{\nu_p}S^2 \cong T_pM$

- shape operator is <u>self-adjoint</u> $\langle S_p v, w\rangle = \langle v, S_p w\rangle \quad \forall\, v, w \in T_pM$

- <u>Second fundamental form</u> quadratic form $w \longmapsto \langle S_p w, w\rangle$

$$h_p = \mathrm{II}_p = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} -\langle \nu_u, x_u\rangle & -\langle \nu_u, x_v\rangle \\ -\langle \nu_v, x_u\rangle & -\langle \nu_v, x_v\rangle \end{pmatrix} = \begin{pmatrix} \langle \nu, x_{uu}\rangle & \langle \nu, x_{uv}\rangle \\ \langle \nu, x_{vu}\rangle & \langle \nu, x_{vv}\rangle \end{pmatrix}$$

- <u>matrix for S_p</u> (w.r.t basis $\{x_u, x_v\}$)

$S_p = g_p^{-1} h_p$

- <u>principal curvatures</u> $\kappa_1\ \kappa_2$ eigenvalues of S_p

<u>principal (curvature) directions</u> e.e. eigenvectors of S_p.

<u>Gauss curvature</u> $K = \kappa_1\kappa_2 = \det S_p = \dfrac{\det h_p}{\det g_p}$

<u>mean curvature</u> $H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\operatorname{tr} S_p$.

<u>mean curvature vector</u> $\vec{H} = H\nu$

- <u>elliptic points</u> $K(p) > 0$ <u>umbilic points</u> $\kappa_1 = \kappa_2$

<u>parabolic points</u> $K(p) = 0$ <u>planar points</u> $\kappa_1 = \kappa_2 = 0$.

<u>hyperbolic points</u>. $K(p) < 0$.

- <u>normal curvature</u> in direction $w \in T_pM$

$$\kappa_n(w) = h_p\left(\frac{w}{|w|}\right) = \frac{\langle S_p w, w\rangle}{\langle w, w\rangle} = \frac{h_p(w)}{g_p(w)}$$ in particular. $|w| = 1 \Rightarrow \kappa_n(w) = h_p(w)$

$w = \cos\theta\, e_1 + \cos\theta\, e_2 \Longrightarrow$ Euler formula $\kappa_n(w) = \cos^2\theta\, \kappa_1 + \sin^2\theta\, \kappa_2$.

principal curvatures are the <u>minimum & maximum</u> of the normal curvatures

- <u>Darboux frame</u>

$\{T, \eta, \nu\}$

tangent, conormal $= \nu \times T$, surface normal

$$\begin{pmatrix} T \\ \eta \\ \nu \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ \eta \\ \nu \end{pmatrix}$$

geodesic curvature, normal curvature, geodesic torsion

$\boxed{3}$

- <u>Curves on surface</u>.

  - curvature line $\quad \tau_g = 0 \quad$ (T principal direction)
  - asymptotic line $\quad \kappa_n = 0 \quad$ (T aysmptotic direction) $\quad \underline{\underline{K \leq 0}}$
  - geodesics $\quad \kappa_g = 0$

some geometric <u>intuition</u>

- along asymptotic lines $\quad$ conormal = principal normal

  Darboux frame = Frenet frame.

- along geodesics $\quad$ surface normal = principal normal.

- straight line $\subseteq M$. $\Rightarrow$ aysmptotic + geodesic
- $M \cap P$ with $P \perp M$ $\Rightarrow$ curvature + geodesic
- $M \cap P$ with P tangent to M $\Rightarrow$ curvature + asymptotic. ($K = 0$).

- <u>Vector field</u>. & <u>flow lines</u> (integral curve, trajectory).

$X: M \longrightarrow \mathbb{R}^3$ s.t. $X_p = X(p) \in T_pM \quad \forall p \in M$.

α curve in M. s.t. $\dot{\alpha} = X$ along α.

<u>Theorem</u>. existence & uniqueness of flow lines. smooth dependence on initial pt.

$\Rightarrow$ (X. Y linearly independent. $\Rightarrow \exists$ parametrization s.t. $x_u \parallel X \quad x_v \parallel Y$).

$\Rightarrow$ existence of orthogonal / asymptotic line / curvature line coordinates.

- <u>First variation of length</u>

$$\delta_\xi \operatorname{len}(\alpha) = \frac{d}{dt}\Big|_{t=0} \operatorname{len}(\alpha_t) = -\int \langle \xi, \vec{k} \rangle \, ds \qquad \alpha^t(s) = \alpha(s) + t\xi(s) + O(t^2)$$

- <u>minimal surfaces</u> $\quad H \equiv 0$

$$\delta_\varphi \operatorname{area}(X) = -2\int_M \varphi H \, dA. \qquad X^t = X + t\varphi\nu$$

- <u>isometry</u> $\quad$ diffeomorphism + <u>local isometry</u> (length-preserving)

$\Updownarrow$

the same first fundamental form.

<u>conformal maps</u>. (angle-preserving).

<u>Thm</u>: all surfaces are locally conformal.

- <u>covariant derivative</u>: tangent part of diretional derivative.

$w \in T_pM$. $\quad Y$ vector field. $\quad \nabla_w Y = (\partial_w Y)^{\parallel} = D_p Y(w) - h(w, Y_p)\nu \in T_pM$.

$\frac{\nabla}{dt} Y := \nabla_{\dot\alpha(t)} Y$. $\quad \alpha$ curve on M.

<u>geodesics</u>: $\boxed{\nabla_T T = 0}$

<u>parallel</u>. <u>parallel transport</u> $\quad \frac{\nabla}{dt} Y = 0$.

X, Y, Z vector fields. $\quad f \in C^\infty(M)$. $\quad c = \text{const.} \in \mathbb{R}$

- $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z \qquad \nabla_- Z$ is $C^\infty(M)$-linear
- $\nabla_X (cY + Z) = c\nabla_X Y + \nabla_X Z \qquad \nabla_X -$ is $\mathbb{R}$-linear
- $\nabla_X (fZ) = (\nabla_X f) Z + f\nabla_X Z \qquad$ Leibniz rule.

$$= (\partial_X f) Z + f\nabla_X Z.$$

- <u>Lie bracket</u> $\quad [X, Y] := \nabla_X Y - \nabla_Y X$.

<u>Thm</u>: $\left(\exists \text{ coordinate such that } X_u = X \;\; X_v = Y\right) \Leftrightarrow \left([X, Y] = 0\right)$

- Christoffel symobols.

$$\nabla_{X_i} X_j = \sum \Gamma_{ij}^k X_k. \qquad \langle X_{ij}, X_k \rangle = \langle \nabla_{X_i} X_j, X_k \rangle = \sum \Gamma_{ij}^l g_{lk} = \Gamma_{ijk}.$$

Gauss formula $\quad X_{ij} = \Gamma_{ij}^1 X_1 + \Gamma_{ij}^2 X_2 + h_{ij} \nu$

lower / raise index.

$$\nabla_X Y = \sum_{i,k} \alpha^i \left(\frac{\partial \beta^k}{\partial u^i} + \sum_j \beta^j \Gamma_{ij}^k \right) X_k. \qquad \begin{array}{l} X = \sum \alpha^i X_i \\ Y = \sum \beta^j X_j \end{array}$$

$$\Gamma_{ij}^k = \frac{1}{2} \sum g^{kl} \left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

- compatublility conditions.

Thm (Bonet) $\quad \left(\begin{array}{l} \text{Gauss equation} \\ \text{Codazzi equation} \end{array} \right) \Leftrightarrow (\exists \text{ surface with given } g, h)$

- ~~Egregium~~ Theorema Egregium: K depends only on g.

in orthogonal coordinates. $\quad K = -\frac{1}{2\sqrt{EG}} \left(\partial_v \left(\frac{E_v}{\sqrt{EG}} \right) + \partial_u \left(\frac{E_u}{\sqrt{EG}} \right) \right)$

conformal coordinates $\quad K = -e^{-2\varphi} \Delta \varphi. \qquad \lambda = e^{\varphi}$. conformal factor.

- Thm (Minding) surfaces with same constant Gauss curvature are locally isom.

Thm (Liebmann) ① Smooth closed surfaces with constant Gauss curvature is a round sphere.

② smoot closed surface with $K > 0$. $H = \text{const.}$ is a round sphere.