TECHNISCHE UNIVERSITÄT BERLIN
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Differential Geometry I - SS 15

http://www3.math.tu-berlin.de/geometrie/Lehre/SS15/DGI/

Homework 9

Problem 1 (1+2+1 points)

Functions $f, g \colon U \in \mathbb{R}^2 \to \mathbb{R}$ satisfying the Cauchy-Riemann equations

$$f_u = g_v, f_v = -g_u,$$

are easily seen to be harmonic; in this situation, we say f and g are *harmonic conjugates*. Let $\mathbf{x}$ and $\mathbf{y}$ be conformal parametrizations of minimal surfaces such that their component functions are pairwise harmonic conjugate; then $\mathbf{x}$ and $\mathbf{y}$ are called *conjugate minimal surfaces*. Show that

(a) Show that the helicoid and catenoid (of Homework 6) are conjugate minimal surfaces.

(b) Given two conjugate minimal surfaces $\mathbf{x}$ and $\mathbf{y}$, show that

$$\mathbf{z}^\theta = \cos\theta\,\mathbf{x} + \sin\theta\,\mathbf{y}$$

is a minimal surface for each $\theta \in \mathbb{R}$.

(c) Show that all these surfaces $\mathbf{z}^\theta$ have the same first fundamental form and hence are locally isometric. (Thus, conjugate minimal surfaces can be joined through a one-parameter family of minimal surfaces with constant first fundamental form.)

Problem 2 (1+2+1 points)

Let $T := T_{R,r}$ denote the torus of revolution with major radius R and minor radius r (as in Homework 8). Let $F \colon \mathbb{R}^2 \to T$ be the map

$$F(u, v) = \big((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u\big).$$

and let $\alpha \colon t \mapsto (at, bt)$ parametrize a straight line in $\mathbb{R}^2$. Consider the curve $F \circ \alpha$ in T.

(a) Show F is a local diffeomorphism.

(b) Show that $F \circ \alpha$ is a regular curve and that it is closed if and only if b/a is rational.

(c) If b/a is irrational, show the image of the curve $F \circ \alpha$ is dense in T; that is, it passes through every neighborhood of every point $p \in T$.

Problem 3 (3 points)

Let $\mathbf{x} = \mathbf{x}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$ be a conformal parametrization with conformal factor $\lambda^2 = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$. Show that

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 2\lambda^2 \vec{H}$$

where $\vec{H}$ is the mean curvature vector. Conclude that in this case $\mathbf{x}$ is a minimal surface if and only if its coordinate functions x, y and z are harmonic.

Problem 4 (4+1 points)

Show that on a *noncylindrical* ruled surface, i.e., a surface patch parametrized by $\mathbf{x}(u, t) = \beta(t) + u\delta(t)$ with $\delta'(t)$ nonvanishing, there exists a *unique* curve $\alpha(t) = \mathbf{x}(\tilde{u}(t), t)$ such that $\langle \delta'(t), \alpha'(t) \rangle = 0$. This curve is called the *line of striction*, and its points are the *central points* of the ruled surface. Show that the helicoid is a ruled surface and that its line of striction is the z-axis.

Homework due in at the start of lecture on Monday 22 June.