

Homework 10

Problem 1

(3 points)

Verify that the surfaces

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, \log u), \quad \mathbf{y}(u, v) = (u \cos v, u \sin v, v)$$

have equal Gauss curvature at corresponding points $\mathbf{x}(u, v)$ and $\mathbf{y}(u, v)$ but that the map $\mathbf{y} \circ \mathbf{x}^{-1}$ is not an isometry. (This shows that the “converse” of the Theorema Egregium doesn’t hold.)

Problem 2

(2+2+3 points)

(a) Let $\mathbf{x}(u^1, u^2)$ be a regular surface and $\alpha(t) = \mathbf{x} \circ \gamma(t) = \mathbf{x}(u^1(t), u^2(t))$ be a curve in the surface. Show that $\alpha(t)$ is a geodesic if and only if $\gamma(t) = (u^1(t), u^2(t))$ satisfies the differential equations

$$\ddot{u}^k(t) + \sum_{i,j} \dot{u}^i(t) \dot{u}^j(t) \Gamma_{ij}^k(\alpha(t)) = 0, \quad k = 1, 2.$$

(b) Compute the Christoffel symbols for the torus $T_{R,r}$ given by

$$\mathbf{x}(u^1, u^2) = ((R + r \cos u^1) \cos u^2, (R + r \cos u^1) \sin u^2, r \sin u^1).$$

(c) The parallels $\mathbf{x}(0, u^2)$, $\mathbf{x}(\pi, u^2)$ and $\mathbf{x}(\pi/2, u^2)$ of the torus $T_{R,r}$ can be called the *maximum parallel*, the *minimum parallel* and the *upper parallel*, respectively. Determine which of these parallels are

- (i) geodesics,
- (ii) asymptotic curves,
- (iii) lines curvature.

Problem 3

(3+1 points)

A ruled surface $\mathbf{x}(t, u) = \beta(t) + u\delta(t)$ with $|\delta(t)| = 1$ is said to be *developable* if the vector triple product

$$[\delta, \delta', \beta'] := \langle \delta \times \delta', \beta' \rangle$$

vanishes for all t . Show that, at regular points, the Gauss curvature of a developable surface is identically zero. Conclude that if $\delta \times \delta' \neq 0$ for all t , then the line of striction (Homework 9) will be the locus of singular points of the developable surface.

Problem 4

(2 points)

Show that there exists no surface with $g_{ij} = \delta_{ij}$ and $h_{ij} = (-1)^i \delta_{ij}$. (Here δ_{ij} is of course the Kronecker delta function, 1 if $i = j$ and 0 if $i \neq j$.)