1 The complex numbers ①

- What are the real numbers?
  - axiomatic characterization
  - constructions
- What are the complex numbers?
  - If F is any field that contains $\mathbb{R}$ and an element i with $i^2 = -1$, then the set of numbers $\mathbb{C} = \{a + bi \,|\, a, b \in \mathbb{R}\}$ is a subfield of F.
  - If $\tilde{\mathbb{C}} = \{a + b\tilde{i}\}$ is another field obtained this way, then there is a unique field isomorphism $\mathbb{C} \to \tilde{\mathbb{C}}$ mapping i to $\tilde{i}$ and every real number to itself.
  - The field of complex numbers is uniquely characterized as the smallest field containing $\mathbb{R}$ and an imaginary unit i.
  - In fact: Every finite dimensional field extension of $\mathbb{R}$ is isomorphic to $\mathbb{C}$. [without proof]
  - Constructive approach: The complex numbers $\mathbb{C}$ are the real vector space $\mathbb{R}^2$ equipped with multiplication
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$
    We write i for $(0, 1)$ and x for $(x, 0)$. From now on, we will use this as the definition of $\mathbb{C}$.
- Collection of some stuff that should be known:
  - Real and imaginary parts, $\operatorname{Re} z$ and $\operatorname{Im} z$
  - complex conjugation $z \mapsto \bar{z}$
  - $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$, $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$
  - polar representation: $z = r(\cos\phi + i\sin\phi)$ where $r > 0$, $\phi \in \mathbb{R}$
  - absolute value (or modulus) $|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{z\bar{z}}$
  - argument ϕ determined up to multiples of 2π (if $z \neq 0$)
  - Euler's formula: $e^{iy} = \cos y + i \sin y$
  - $e^{x+iy} = e^x(\cos y + i\sin y)$
  - $z = e^{\log|z| + i\phi}$
  - geometric interpretation of complex multiplication
  - complex multiplication as $\mathbb{R}$-linear map

Literature Many good books cover the material of this course. I mostly follow

- Ahlfors. *Complex Analysis*
- Jänich. *Funktionentheorie*
- Dirk Ferus' lecture notes, http://page.math.tu-berlin.de/~ferus/KA/komplexeAnalysis.pdf

2 Complex differentiation

Definition 2.1. Let $U \in \mathbb{C}$ be an open subset, $z_0 \in U$. A function $f : U \to \mathbb{C}$ is called *[complex] differentiable* in z_0, if the limit
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0)$$
exists. In this case, $f'(z_0)$ is called the *[complex] derivative* of f at z_0. If f is differentiable at all points $z_0 \in U$, then f is called *holomorphic* (or *[complex] analytic*) on U. A holomorphic function on $\mathbb{C}$ is called an *entire function [ganze Funktion]*.

- same definition of differentiability and derivative as in Analysis I, only $\mathbb{R}$ replaced by $\mathbb{C}$

- Basic rules for differentiation remain valid, with same proofs
- Examples:
  - Polynomials $p(z) = \sum_{i=0}^{n} a_i z^i$ are entire functions
  - Rational functions $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials and $q \neq 0$, are holomorphic on $U = \{z \,|\, q(z) \neq 0\}$.
  - power series (see following section)

3 Power series

(2)

Theorem 3.1. *(i) For every power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ (with complex coefficients a_k) there is an $R \in [0, \infty]$ (the radius of convergence), such that the series converges absolutely if $|z - z_0| < R$ and diverges if $|z - z_0| > R$.*

(ii) In general,

$$R^{-1} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}\,.$$

(iii) If the limit on the right hand side exists, then

$$R^{-1} = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}\,.$$

(iv) If $0 < r < R$, then the power series converges uniformly on the closed disk $\{z \,\big|\, |z - z_0| \leq r\}$.

Proof. Exactly like in real analysis. □

Theorem 3.2. *If $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ has radius of convergence $R > 0$, then f is holomorphic on the open disk with radius R around z_0, and one can differentiate term by term: The power series $\sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$ also has radius of convergence R and represents $f'(z)$.*

Proof. Later, using complex integration. □

- Examples
  - e^z
  - $\cos z$, $\sin z$
  - $\cosh z$, $\sinh z$

4 Real and complex differentiability

The function $f : U \to \mathbb{C}$ is complex differentiable at z_0 with derivative $f'(z_0)$ if and only if

$$f(z_0 + h) = f(z_0) + f'(z_0)h + r(h) \quad \text{with} \quad \lim_{h \to 0} \frac{r(h)}{h} = 0.$$

The function f is differentiable the real sense at z_0 if

$$f(z_0 + h) = f(z_0) + df_{z_0}(h) + r(h) \quad \text{with} \quad \lim_{h \to 0} \frac{r(h)}{\|h\|} = 0.$$

Theorem 4.1. *The function $f(x, y) = \big(u(x, y), v(x, y)\big)$ is complex differentiable in z_0 if and only if it is differentiable in the real sense and one (hence both) of the following equivalent conditions hold:*

(i) The real derivative

$$df = \begin{pmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{pmatrix}$$

is $\mathbb{C}$-linear at z_0.

(ii) The Cauchy–Riemann equations

$$\frac{du}{dx} = \frac{dv}{dy}, \qquad \frac{du}{dy} = -\frac{dv}{dx}$$

are satisfied at z_0.

Example 4.2. $f(z) = \bar{z}$

Theorem 4.3. *If f is holomorphic on an open and connected set $U \subseteq \mathbb{C}$, and if $f'(z) = 0$ for all $z \in U$, then f is constant.*

Definition 4.4. An open and connected subset of $\mathbb{C}$ is called a *domain*.

Theorem 4.5. *If f is holomorphic and real valued on a domain, then f is constant.*

Important Example 4.6.

- *Principal value argument function*, $\operatorname{Arg} : \mathbb{C} \setminus (-\infty, 0] \to (-\pi, \pi)$.
- *Principal value logarithm function*, $\operatorname{Log} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}, \quad \operatorname{Log}(z) = \log|z| + i \operatorname{Arg} z$.

Harmonic functions ③

- Laplace operator $\Delta = \operatorname{div}\operatorname{grad} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$
- Laplace equation $\Delta f = 0$
- harmonic functions

Theorem 4.7. *Real and imaginary parts of a holomorphic [and twice continuously differentiable] function are harmonic.*

Remark. We will see later that holomorphic functions are C^∞, so the condition in brackets is automatically satisfied.

Theorem 4.8. *If $u : U \to \mathbb{R}$ is a harmonic function on a convex/star shaped/simply connected domain $U \subseteq \mathbb{C}$, then there exists a harmonic function $v : U \to \mathbb{R}$ so that $f = u + iv$ is holomorphic. The function v is uniquely determined up to an additive constant.*

Functions u and v that are the real and imaginary part of a holomorphic function are called *conjugate harmonic functions*.

Theorem 4.9. *If $h : U \to \mathbb{R}$ is harmonic and f is a holomorphic function mapping $\tilde{U}$ onto U, then the composition $h \circ f$ is harmonic on $\tilde{U}$.*

5 Conformal maps

Definition 5.1. A *conformal map* is a map that preserves angles and orientation.

Theorem 5.2. *A holomorphic function whose derivative has no zeros is a conformal map.*

Theorem 5.3. *Suppose a map $f : U \to \mathbb{C}$ of a domain $U \subseteq \mathbb{C}$ is differentiable in the real sense and conformal. Then f is holomorphic.* ④

- *stereographic projection*: conformal and circle preserving
- *Riemann sphere*
- *extended complex plane*: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with topology inherited from S^2 via stereographic projection
- Mercator projection

6 Möbius transformations

Definition 6.1. A Möbius transformation of the complex plane is a function of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad \text{where} \quad ad - bc \neq 0.$$

- W.l.o.g., one may assume $ad - bc = 1$.
- similarity transformations $z \mapsto az + b$
- inversion $z \mapsto \frac{1}{z}$ (geometric interpretation)
- inverse, composition

Theorem 6.2. *The Möbius transformations form a group of maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the Möbius group $\mathscr{M}$. The map*

$$SL(2,\mathbb{C}) \to \mathscr{M}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \mapsto \frac{az+b}{cz+d} \right)$$

is a group homomorphism with kernel $\{+1,-1\}$. The Möbius group $\mathscr{M}$ is therefore isomorphic to $SL(2,\mathbb{C})/\{+1,-1\}$.

Theorem 6.3. *Every Möbius transformation is either a similarity transformation or the composition of a similarity transformation, the inversion $z \mapsto \frac{1}{z}$, and a similarity transformation.*

Theorem 6.4. *A Möbius transformation maps circles to circles. (Straight lines are considered as circles through ∞.)*

- *Warning*: The center of a circle is in general *not* mapped to the center of the image circle. ⑤

Theorem 6.5. *For three different points z_1, z_2, z_3 in $\hat{\mathbb{C}}$ and three different points w_1, w_2, w_3, there is a unique Möbius transformation f with $f(z_k) = w_k$, $k = 1,2,3$.*

Example 6.6. The Möbius transformation

$$f(z) = \frac{z-i}{z+i}$$

maps the real axis (incl. ∞) to the unit circle and the upper half plane $H = \{z \mid \operatorname{Im} z > 0\}$ to the unit disk $D = \{z \mid |z| < 1\}$.

Definition 6.7. The *cross-ratio* of four different points z_1, z_2, z_3, z_4 in $\hat{\mathbb{C}}$ is

$$\operatorname{cr}(z_1,z_2,z_3,z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} \tag{1}$$

Equivalently, $\operatorname{cr}(z_1,z_2,z_3,z_4)$ is the image of z_1 under the Möbius transformation that maps z_2, z_3, z_4 to 0, 1, ∞, respectively.

- If one of the four points is ∞, the right hand side of equation (1) is extended continuously, by sensibly "cancelling infinities". For example,

$$\operatorname{cr}(z_1,z_2,\infty,z_4) = \frac{(z_1-z_2)\cancel{(\infty-z_4)}^{-1}}{\cancel{(z_2-\infty)}(z_4-z_1)} = -\frac{z_1-z_2}{z_4-z_1}.$$

Warning: Definitions with different permutations of the arguments are common in the literature.

Theorem 6.8. *Four different points lie on a circle if and only if their cross-ratio is real.*

Theorem 6.9. *For every Möbius transformation and any four different points z_1, z_2, z_3, z_4,*

$$\operatorname{cr}\big(f(z_1),f(z_2),f(z_3),f(z_4)\big) = \operatorname{cr}(z_1,z_2,z_3,z_4).$$

Theorem 6.10. *(i) A Möbius transformations f maps the upper half plane*

$$H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

onto itself if and only if there are real coefficients a, b, c, d with $ad-bc > 0$ (or, equivalently, $ad-bc = 1$) such that $f(z) = \frac{az+b}{cz+d}$. ⑥

(ii) A Möbius transformation f maps the unit disk D onto D if and only if one, and hence both, of the following conditions are satisfied:

- *There are $a, b \in \mathbb{C}$ such that $|a|^2 - |b|^2 > 0$ (or, equivalently, $|a|^2 - |b|^2 = 1$) and*

$$f(z) = \frac{az+\bar{b}}{bz+\bar{a}}.$$

- *There is a number $\phi \in \mathbb{R}$ and a point $z_0 \in D$ such that*

$$f(z) = e^{i\phi}\,\frac{z-z_0}{1-\bar{z}_0 z}.$$

7 Complex integration

- If $h : [a,b] \to \mathbb{R}^n$ is continuous, then the integral $\int_a^b h(t)\,dt$ is defined component-wise as Riemann or Lebesgue integral.

Definition 7.1 (contour integral)**.** Let $U \subseteq \mathbb{C}$ be an open subset, let $f : U \to \mathbb{C}$ be a continuous function, and let $\gamma : [a,b] \to U$ be continuously differentiable. Then

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

is called the *[contour] integral* of the function f along the curve γ. If γ is only piecewise continuously differentiable, then contour integral is the sum of integrals, as defined above, over the sub-intervals where γ is differentiable.

- The contour integral is invariant under re-parameterization of the curve γ. More precisely, if $\phi : [c,d] \to [a,b]$ is continuously differentiable and $\phi(c) = a$, $\phi(d) = b$, then

  $$\int_\gamma f(z)\,dz = \int_{\gamma\circ\phi} f(z)\,dz.$$

  If $\phi(c) = b$, $\phi(d) = a$, then the integral changes sign:

  $$\int_\gamma f(z)\,dz = -\int_{\gamma\circ\phi} f(z)\,dz.$$

- If f is bounded by $C \in \mathbb{R}$ (i.e., $|f(z)| \leq C$ for all $z \in U$) then

  $$\left|\int_\gamma f(z)\,dz\right| \leq C\,\text{length}(\gamma).$$

(7)

Theorem 7.2 (Fundamental theorem of complex calculus)**.** *Let $U \subseteq \mathbb{C}$ be an open subset, and let $f : U \to \mathbb{C}$ be a continuous function. Then the following conditions are equivalent:*

(i) For all closed *paths η in U,*

$$\int_\eta f(z)\,dz = 0.$$

(ii) The integral of f along a path $\gamma : [a,b] \to U$ depends only on the endpoints $\gamma(a)$, $\gamma(b)$.
(iii) There is a holomorphic function $F : U \to \mathbb{C}$ with $F' = f$.

If one, and hence all of these statements are true, then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Example 7.3. $\int_\gamma z^n\,dz$, where $\gamma : [0,2\pi] \to \mathbb{C}$, $\gamma(t) = re^{it}$, $r > 0$.

- *Notation:* If γ is a circle around z_0 with radius r, i.e.,

  $$\gamma : [0,2\pi] \to \mathbb{C}, \quad \gamma(t) = z_0 + re^{it},$$

  one writes $\int_{|z-z_0|=r} \ldots dz$ for $\int_\gamma \ldots dz$.

Theorem 7.4 (Integration of power series). *Suppose the power series $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ has radius of convergence $R > 0$. Then the power series $F(z) = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(z - z_0)^{k+1}$ also has radius of convergence R, and $F' = f$.* ⑧

Theorem 7.5 (Differentiation of power series). *A power series $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ with radius of convergence $R > 0$ is holomorphic on $\{z \mid |z - z_0| < R\}$ and the derivative is $f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$, which is a power series with the same radius of convergence R.*

8 Vector analysis interpretation of the contour integral

Let $U \in \mathbb{R}^2 = \mathbb{C}$ be an open subset, let $v = v_1 + i v_2 : U \to \mathbb{C}$ be a vector field, and let

$$f = \bar{v}.$$

Then, for a piecewise C^1 curve $\gamma : [a, b] \to U$,

$$\int_\gamma f(z) dz = \underbrace{\int_a^b \langle v(\gamma(t)), \gamma'(t) \rangle \, dt}_{\text{integral of } v \text{ along } \gamma} + i \underbrace{\int_a^b \det\big(v(\gamma(t)), \gamma'(t)\big) \, dt}_{\text{flow of } v \text{ through } \gamma}.$$

Now suppose γ is the boundary curve of a bounded domain $G \in C$. Then the classical integral theorems of Gauss and Stokes (2D version) say

$$\text{flow of } v \text{ through } \gamma = \int_G \operatorname{div}(v) \, dx_1 \, dx_2, \qquad \text{(Gauss)}$$

$$\text{integral of } v \text{ along } \gamma = \int_G \operatorname{rot}(v) \, dx_1 \, dx_2, \qquad \text{(2D Stokes)}$$

where

$$\operatorname{div}(v) = \partial_1 v_1 + \partial_2 v_2, \qquad \operatorname{rot}(v) = \partial_1 v_2 - \partial_2 v_1.$$

The Cauchy–Riemann-equations for f are equivalent to $\operatorname{div} v = \operatorname{rot} v = 0$.

Theorem 8.1 (Cauchy's integral theorem, vector analysis version). *Let f be holomorphic on U with continuous derivative f', and let the curve γ in U be the piecewise C^1 boundary curve of a bounded domain $G \in C$. Then*

$$\int_\gamma f(z) \, dz = 0.$$

9 Cauchy's integral theorem

Theorem 9.1 (Cauchy's integral theorem for rectangles). *Let f be holomorphic in U (by definition, this implies that U is open in $\mathbb{C}$), let*

$$Q = \{z \mid \operatorname{Re} z \in [a, b], \operatorname{Im} z \in [c, d]\}$$

be a closed rectangle contained in U, and let γ be a piecewise continuously differentiable parameterization of the boundary curve of Q. Then

$$\int_\gamma f(z) \, dz = 0.$$

Theorem 9.2 (Cauchy's integral theorem for C^1-images of rectangles)**.** *Let f be holomorphic on U, let $Q = [a,b] + i[c,d]$ be a closed rectangle in $\mathbb{C}$, let γ be a piecewise C^1 parameterization of the boundary of Q, and let $\phi : Q \to \mathbb{C}$ be a continuously differentiable function such that $\phi(Q) \subset U$. Then* ⑨

$$\int_{\phi\circ\gamma} f(z)\,dz = 0.$$

- "Remember": If $W \subseteq \mathbb{R}^m$ is open, then the derivative of a function $\phi : W \to \mathbb{R}^n$ is defined in terms of approximating linear functions. The definition is extended to functions ϕ whose domain of definition W is not open by assuming that ϕ can be extended to a differentiable function on an open neighborhood of W.
- Corollaries:
  - ◇ Cauchy's integral theorem for triangles
  - ◇ Cauchy's integral theorem for disks
  - ◇ Cauchy's integral theorem for annuli [Kreisringe]
  - ◇ Cauchy's integral theorem for nested non-concentric circles
  - ◇ Cauchy's integral theorem for C^1-homotopic curves (→ homework?)
  - ◇ Cauchy's theorem for freely C^1-homotopic closed curves (→ homework?)

10 Cauchy's integral formula and the mean value theorem

Theorem 10.1 (Cauchy's integral formula for a disk)**.** *Let f be holomorphic on a domain that contains the closed disk $\{z \in \mathbb{C} \mid |z - z_0| \le r\}$. Then, for all points a in the interior of the disk,*

$$f(a) = \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f(z)}{z-a}\,dz.$$

Theorem 10.2 (Mean value theorem)**.** *Let f be holomorphic on a domain that contains the closed disk $\{z \in \mathbb{C} \mid |z - z_0| \le r\}$. Then the value of f at the center of the disk is the mean of the values on the circle $|z - z_0| = r$, i.e.,* ⑩

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\phi})\,d\phi.$$

11 Power series theorem and some consequences

Theorem 11.1 (power series theorem)**.** *Let f be holomorphic on U and suppose U contains the open disk $D_{z_0,R}$ with center z_0 and radius $R > 0$. Then, for all $z \in D_{z_0,R}$,*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

with arbitrary $r \in (0, R)$.

Corollary 11.2. *Holomorphic functions are arbitrarily often differentiable.*

In particular, holomorphic functions are continuously differentiable, and the real inverse function theorem for continuously differentiable functions implies the complex inverse function theorem:

Corollary 11.3 (Inverse function theorem). *Let f be holomorphic on U, and let $z_0 \in U$ with $f'(z_0) \neq 0$. Then there is an open neighborhood $U_0 \subseteq U$ of z_0 such that f is injective on U_0, $f(U_0)$ is open, and the inverse $(f|_{U_0})^{-1}$ is holomorphic on U_0.*

Corollary 11.4 (Cauchy integral formula for derivatives). *Let f be holomorphic on a domain that contains the closed disk $\{z \in \mathbb{C} \mid |z - z_0| \leq r\}$. Then, for all points w in the interior of the disk,*

$$f^{(k)}(w) = \frac{k!}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f(z)}{(z-w)^{k+1}}\, dz.$$

Corollary 11.5 (Estimate for Taylor coefficients). *Let $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ be a power series with radius of convergence $R \in (0, \infty]$. Suppose $0 < r < R$ and $|f(z)| \leq M$ for all z with $|z - z_0| \leq r$. Then for all $k \in \{0, 1, 2, \dots\}$*

$$|a_k| \leq \frac{M}{r^k}.$$

Corollary 11.6 (Liouville's theorem). *A bounded entire function is constant.*

Corollary 11.7 (Fundamental theorem of algebra). *Every non-constant complex polynomial has a zero.*

- Using polynomial division, one deduces that the number of zeros (counting multiplicities) of a non-zero polynomial is equal to the degree.

12 Zeros of holomorphic functions 11

Definition 12.1 (Order of a zero). A holomorphic function f has a *zero of order n* in a, if

$$\begin{aligned} f^{(k)}(a) &= 0 \qquad \text{if } 0 < k < n, \\ f^{(n)}(a) &\neq 0. \end{aligned}$$

Proposition 12.2 (Zero of infinite order). *If f is holomorphic on a domain U, and $f^{(k)}(a) = 0$ for some $a \in U$ and for all $k \in \mathbb{Z}_{\geq 0}$, then $f = 0$.*

Proposition 12.3. *A holomorphic function f on U has a zero of order n at $a \in U$ if and only if*

$$f(z) = (z - a)^n g(z)$$

for some holomorphic function g on U with $g(a) \neq 0$.

Theorem 12.4. *If f has a zero of order n at z_0, then there is an open neighborhood U_0 of z_0 and an holomorphic function h on U_0 with a simple zero at z_0 such that*

$$f(z) = h(z)^n.$$

Theorem 12.5 (Behavior of a holomorphic function near a zero). *If f has a zero of order n at z_0 and if $\epsilon > 0$ is small enough, then there is an open neighborhood $U_\epsilon \ni z_0$ such that*

- *f maps U_ϵ onto the ϵ-disk $\{w \mid |w| < \epsilon\}$,*
- *f attains every value w with $0 < |w| < \epsilon$ exactly n times in U_ϵ,*
- *f attains the value 0 in U_0 only at z_0.*

Corollary 12.6. *Zeros of finite order of a holomorphic function are isolated.*

13 Identity theorem and invariance of domain

Theorem 13.1 (Identity theorem). *Let f and g be holomorphic functions on a domain U, and suppose $f(z) = g(z)$ for all z in a subset of U that has an accumulation point in U. Then $f = g$.*

Theorem 13.2 (Invariance of domain). *If f is holomorphic an not constant on a domain U, then $f(U)$ is also a domain.* (12)

Theorem 13.3 (Maximum principle). *If f is holomorphic and not constant on a domain U, then the real valued functions $|f|$, $\operatorname{Re} f$, $\operatorname{Im} f$ do not attain a maximum.*

Corollary 13.4 (Maximum principle, boundary version). *If f is holomorphic on a domain U, and if $A \subseteq U$ is a compact subset, then $|f|$, $\operatorname{Re} f$, $\operatorname{Im} f$ attain their maximum on the boundary of A.*

Theorem 13.5 (Schwartz's Lemma). *If f is a holomorphic function on the unit disk D with image $f(D) \subseteq D$, and $f(0) = 0$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all z. If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then f is rotation: $f(z) = e^{i\alpha} z$ for some $\alpha \in \mathbb{R}$.*

Theorem 13.6 (Automorphisms of disk and half-plane). *(i) Suppose f is an injective holomorphic map on the unit disk D with image $f(D) = D$. Then f is a Möbius transformation mapping D onto D.*

(ii) Suppose f is an injective holomorphic map on the upper half-plane H with image $f(H) = H$. Then f is a Möbius transformation mapping H onto H.

Definition 13.7. An injective holomorphic map with holomorphic inverse is called *biholomorphic*.

Theorem 13.8. *An injective holomorphic map is biholomorphic.*

14 Morera's theorem and the reflection principle

Theorem 14.1 (Morera). *Let $f : U \to \mathbb{C}$ be a continuous function on an open subset $U \subseteq \mathbb{C}$. If*

$$\int_\gamma f(z)\,dz = 0$$

whenever γ is the boundary curve of a closed triangular region contained in U, then f is holomorphic. (13)

Theorem 14.2 (Schwarz reflection principle). *Let $U \subseteq \mathbb{C}$ be an open subset which is symmetric with respect to reflection on the real axis, that is, $z \in U \Longleftrightarrow \bar{z} \in U$. Suppose the function*

$$f : U \cap \{z \mid \operatorname{Im} z \geq 0\} \longrightarrow \mathbb{C}$$

is continuous, holomorphic on $U \cap \{z \mid \operatorname{Im} z > 0\}$ and real valued on $U \cap \mathbb{R}$. Then the function

$$\hat{f} : U \to \mathbb{C}, \qquad \hat{f}(z) = \begin{cases} f(z) & \text{if } \operatorname{Im} z \geq 0 \\ \overline{f(\bar{z})} & \text{if } \operatorname{Im} z < 0 \end{cases}$$

is holomorphic.

15 Isolated singularities

Definition 15.1 (isolated singularity). A point $z_0 \in \mathbb{C}$ is called an *isolated singularity* of a holomorphic function f on $U \subseteq \mathbb{C}$ if $z_0 \notin U$, but there is an $\epsilon > 0$ such that

$$\{z \in \mathbb{C} \,|\, 0 < |z - z_0| < \epsilon\} \subseteq U.$$

Definition 15.2 (removable singularity). An isolated singularity z_0 of a holomorphic function f on U is called *removable* if there is a holomorphic function $\hat{f}$ on $U \cup \{z_0\}$ with $\hat{f}|_U = f$.

Theorem 15.3 (Riemann's theorem on removable singularities). *If z_0 is a removable singularity of the holomorphic function f, then the following statements are equivalent:*

(i) *z_0 is a removable singularity.*

(ii) *f is bounded in a neighborhood of z_0. That is, there is an $\epsilon > 0$ such that the restriction of f to*

$$\left\{z \in \mathbb{C} \,\middle|\, 0 < |z - z_0| < \epsilon\right\}$$

is bounded.

(iii) $\lim\limits_{z \to z_0} (z - z_0) f(z) = 0$

Definition and Theorem 15.4 (Three types of isolated singularities). Let z_0 be an isolated singularity of f. Then exactly one of the following statements is true:

(i) $f(z)$ is bounded in a neighborhood of z_0 and z_0 is a removable singularity.

(ii) $\lim_{z \to z_0} |f(z)| = \infty$. Then z_0 is called a *pole* and $(z - z_0)^m f(z)$ has a removable singularity for some $m \in \mathbb{Z}_{>0}$. The smallest m with this property is called the *order* of the pole.

(iii) Neither (i) nor (ii) are true. Then z_0 is called an *essential singularity*.

Examples 15.5. (i) $f(z) = \frac{1}{1 - z^2}$

(ii) $g(z) = \frac{1}{\sin z}$

(iii) $h(z) = \frac{z}{\sin z}$

(iv) $q(z) = e^{1/z}$

Theorem 15.6 (Cassorati–Weierstrass). *Let z_0 be an essential singularity of $f : U \to \mathbb{C}$ and let $\epsilon > 0$* (14)
be small enough so that $U_0 = \{z \,|\, 0 < |z - z_0| < \epsilon\} \subseteq U$. Then $f(U_0)$ is dense in $\mathbb{C}$.

- In other words, the conclusion of Theorem 15.6 says the following: For every $w \in \mathbb{C}$, every $\delta > 0$, and every $\epsilon > 0$, there is a $z \in U$ with $|z - z_0| < \epsilon$ such that $|f(z) - w| < \delta$.
- In fact, a stronger statement is true:

Theorem 15.7 (Great Picard's theorem). *In every neighborhood of an essential singularity, a holomorphic function attains every value in $\mathbb{C}$ with at most one exception.* [no proof given]

Definition 15.8. Let $U \subseteq \mathbb{C}$ be an open subset. A function is called *holomorphic on U up to isolated singularities* if f is holomorphic on $U \setminus S$ for some subset $S \subseteq U$ and all points of S are isolated singularities. If all the isolated singularities are poles, then f is called *meromorphic*.

Proposition 15.9. *If f and g are holomorphic on U and g is not constantly zero, then $\frac{f}{g}$ is meromorphic on U (after all removable singularities are removed).*

Remark 15.10. The set of meromorphic functions on an open subset $U \subseteq \mathbb{C}$ is denoted by $\mathcal{M}(U)$. If $+$ and $\cdot$ are defined on $\mathcal{M}(U)$ by first adding/multiplying functions on the intersection of their domains, then removing all removable singularities, then $\mathcal{M}(U)$ is a ring with the constant functions 0 and 1 as 0-element and 1-element, respectively. If U is a domain, then $\mathcal{M}(U)$ is a field.

16 Laurent series

A *Laurent series* is a series of the form

$$\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k. \tag{2}$$

More precisely, a *Laurent series* is a pair of two series

$$\underbrace{\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}}_{\textit{principal part}} \quad \text{and} \quad \underbrace{\sum_{k=0}^{\infty} a_k (z-z_0)^k}_{\textit{holomorphic part}}.$$

The holomorphic part is a normal power series with some radius of convergence $0 \le R \le \infty$. The principal part is a power series in $\frac{1}{z-z_0}$. Let $\frac{1}{r}$ be its radius of convergence. If $0 \le r < R \le \infty$, then both series converge on the annulus

$$A = \{z \in \mathbb{C} \mid r < |z-z_0| < R\}. \tag{3}$$

In this case the Laurent series is said to converge on A, and $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ denotes the sum of the limits of both series, the principal and the holomorphic part. Both parts converge absolutely on A and uniformly on compact subsets of A.

Laurent series can be differentiated term by term. One can take antiderivatives term by term as long as $a_{-1} = 0$. (Both statements can be seen by substituting $\zeta = \frac{1}{z-z_0}$ in the principal part.)

Theorem 16.1 (Uniqueness of Laurent series). *Suppose the Laurent series* $f(z) = \sum_{k=-\infty}^{\infty}(z-z_0)^k$ *converges on the annulus* (3). *Then for all* $k \in \mathbb{Z}$ *and any* $\rho \in (r, R)$,

$$a_k = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{k+1}}\, dz. \tag{4}$$

In particular, the integral does not depend on ρ.

Lemma 16.2 (Cauchy formula for annuli). *Let* $A = \{z \mid r < |z-z_0| < R\} \neq \emptyset$ *and let* f *be holomorphic* 15
on a domain containing A. Then, for $z \in A$ *and* ρ_1, ρ_2 *satisfying* $r < \rho_1 < |z-z_0| < \rho_2 < R$,

$$f(z) = \frac{1}{2\pi i}\int_{|u-z_0|=\rho_2} \frac{f(u)}{u-z}\, du - \frac{1}{2\pi i}\int_{|u-z_0|=\rho_1} \frac{f(u)}{u-z}\, du\,.$$

Theorem 16.3 (Laurent series theorem). *Let* $A = \{z \mid r < |z-z_0| < R\} \neq \emptyset$ *and let* f *be holomorphic on a domain containing A. Then, for all* $z \in A$,

$$f(z) = \sum_{k-\infty}^{\infty} a_k(z-z_0)^k,$$

where a_k *defined by* (4) *with arbitrary* $\rho \in (r, R)$.

- Suppose f has an isolated singularity at z_0. Then f has a convergent Laurent series on $0 < |z-z_0| < R$ for some $R > 0$. The principal part of this Laurent series is also called the *principal part of f in z_0*.
- There are three possibilities:
  - ⋄ $a_k = 0$ for all $k < 0$. Then z_0 is removable.
  - ⋄ $a_k \neq 0$ some but only finitely many $k < 0$. Then the Laurent series is $\sum_{k=-n}^{\infty} a_k(z-z_0)^k$ with $a_{-n\neq 0}$, and z_0 is a pole of order n.
  - ⋄ $a_k \neq 0$ for infinitely many $k < 0$. Then z_0 is an essential singularity.

Example 16.4. Laurent series of $f(z) = \frac{1}{1-z}$ in $0 < |z-1|$, $|z| < 1$ and in $1 < |z|$.

17 Analytic continuation ⑯

Definition 17.1. A *function element* is a pair (f, U) of a domain $U \subseteq \mathbb{C}$ and a holomorphic function f on U. Two function elements (f, U) and (g, V) are called *equivalent at a point* $a \in \mathbb{C}$, written

$$(f, U) \overset{a}{\sim} (g, V),$$

if $a \in U \cap V$ and $f(z) = g(z)$ in a neighborhood of a.

- The relation $\overset{a}{\sim}$ is an equivalence relation on the set of function elements.

Definition 17.2 (analytic continuation)**.** A function element (g, V) is called an *analytic continuation* of a function element (f, U), if there is a sequence of function elements

$$(f, U) = (f_0, U_0), \quad (f_1, U_1), \quad \dots, \quad (f_n, U_n) = (g, V)$$

and a sequence of points $a_0, \dots, a_{n-1}$, such that

$$(f_k, U_k) \overset{a_k}{\sim} (f_{k+1}, U_{k+1})$$

for all $k \in \{0, \dots, n-1\}$.

Proposition 17.3 (algebraic and differential equations)**.** *Let a_k and b be entire functions for $k \in \{0, \dots, n\}$.*

(i) If a function element (f, U) satisfies the equation

$$a_n(z)(f(z))^n + \dots + a_1(z)f(z) + a_0(z) = 0$$

for all $z \in U$, then any analytic continuation (g, V) also satisfies

$$a_n(z)(g(z))^n + \dots + a_1(z)g(z) + a_0(z) = 0$$

for all $z \in V$.

(ii) If a function element (f, U) satisfies the differential equation

$$a_n(z)f^{(n)}(z) + \dots + a_1(z)f'(z) + a_0(z)f(z) = b(z)$$

on U, then any analytic continuation (g, V) satisfies

$$a_n(z)g^{(n)}(z) + \dots + a_1(z)g'(z) + a_0(z)g(z) = b(z)$$

on V.

Examples 17.4. • $f(z)^2 = z$
• $f'(z) = \frac{1}{z}$

Definition 17.5 (analytic continuation along a curve)**.** Let $c : [0, 1] \to \mathbb{C}$ be a continuous curve. An *analytic continuation along* c is a family $(f_t, U_t)_{t \in [0,1]}$ of function elements, such that

(i) For all $t \in [0, 1]$, $c(t) \in U_t$.
(ii) For every $t \in [0, 1]$, there is a $\delta > 0$ such that for all $s \in [0, 1]$ with $|s - t| < \delta$

$$(f_t, U_t) \overset{c(s)}{\sim} (f_s, U_s).$$

In this case, one says that (f_1, U_1) is an *analytic continuation* of (f_0, U_0) *along the curve* c.

- A function element (f_1, U_1) is an analytic continuation of (f_0, U_0) (by Definition 17.2) if and only if there is a curve c such that (f_1, U_1) is an analytic continuation of (f_0, U_0) along c (by Definition 17.5).
- It is obvious how to define analytic continuation along curves with parameter interval $[a, b]$ other than $[0, 1]$.

Example 17.6 (trivial continuation)**.** Let f be holomorphic on a domain U, and let $c : [0, 1] \to U$ be a curve in U. Then $(f_t, U_t) = (f, U)$ is an analytic continuation along c.

Example 17.7 (logarithm)**.** Let $c : [0, 1] \to \mathbb{C} \setminus \{0\}$ be a C^1-curve with $c(0) = 1$. For each $t \in [0, 1]$, there is an antiderivative L_t of $z \mapsto \frac{1}{z}$, defined on the open disk

$$D_t = \left\{ z \,\middle|\, |z - c(t)| < |c(t)| \right\},$$

and with

$$L_t(c(t)) = \int_{c|_{[0,t]}} \frac{1}{z}\, dz.$$

In particular, L_0 is the principal value logarithm function restricted to D_0.

The family $(L_t, D_t)_{t \in [0,1]}$ is an analytic continuation along c.

Lemma 17.8 (continuation and power series)**.** *Let (f_t, U_t) be an analytic continuation along a curve $c : [0, 1] \to \mathbb{C}$. For each $t \in [0, 1]$, let $\hat{f}_t$ be the holomorphic function on D_t represented by the power series expansion of f_t around $c(t)$, where D_t is the disk of convergence of the power series. Then $(\hat{f}_t, D_t)_{t \in [0,1]}$ is also an analytic continuation along c, and clearly $(f_t, U_t) \overset{c(t)}{\sim} (\hat{f}_t, D_t)$ for all $t \in [0, 1]$.*

Theorem 17.9 (Uniqueness)**.** *Let (f_t, U_t) and (g_t, V_t) be two analytic continuations along a curve* 17
$c : [0, 1] \to \mathbb{C}$. Then

$$(f_0, U_0) \overset{c(0)}{\sim} (g_0, V_0) \Longrightarrow (f_1, U_1) \overset{c(1)}{\sim} (g_1, V_1).$$

Lemma 17.10 (Lebesgue number)**.** *Let $(U_j)_{j \in J}$ be an open cover of a compact metric space X. Then there is an $\epsilon > 0$ such that for every subset $A \subseteq X$ with $\operatorname{diam}(A) < \epsilon$, there is a $j \in J$ with $A \subseteq U_j$.*

Lemma 17.11. *Let (f_t, U_t) be an analytic continuation along a curve $c : [0, 1] \to \mathbb{C}$. Then there is a subdivision*

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

of $[0, 1]$ and a sequence of function elements $(g_k, D_k)_{1 \le k \le n-1}$, where D_k are disks, such that

$$(f_t, U_t) \overset{c(t)}{\sim} (g_k, D_k) \quad \text{for all} \quad t \in [t_{k-1}, t_{k+1}].$$

Lemma 17.12. *Let $c : [0, 1] \to \mathbb{C}$ be a continuous curve, and let U_0 be an open neighborhood of $c(0)$. A function element (f_0, U_0) can be continued analytically along c if and only if the derived function element $(f'(0), U_0)$ can be continued analytically along c.*

Definition 17.13 (Integration along continuous curves)**.** Let f be a holomorphic function on U and let $c : [0, 1] \to U$ be a continuous curve. We define

$$\int_c f(z)\, dz := F_1(c(1)) - F_0(c(0)), \tag{5}$$

where F_0 and F_1 are defined as follows: For a small disk U_0 around $c(0)$, the function $f|_{U_0}$ has an antiderivative F_0. Since $(f, U_0) = (F', U_0)$ can be continued (trivially) along c, so can (F_0, U_0). Let $(F_t, U_t)_{t \in [0,1]}$ be such an analytic continuation.

Proposition 17.14. *If c is piecewise C^1, the integral of Definition 17.13 has the same value as the integral of Definition 7.1.*

Theorem 17.15 (Cauchy's integral theorem for continuous images of rectangles). *Let f be holomorphic on U, let $Q = [0,1] \times [0,1]$, let $\phi : Q \to U$ be a continuous function, let $\hat{\gamma} : [0,1] \to \mathbb{R}^2$ be a parametrization of the boundary ∂Q, and let $\gamma = \phi \circ \gamma$. Then* (18)

$$\int_\gamma f(z)\, dz = 0.$$

18 Homotopy[1]

In the following $U \subseteq \mathbb{C}$ is an open subset. But most definitions and theorems remain valid if U is any topological space.

Definition 18.1 (homotopic curves). Two curves $c_0, c_1 : [0,1] \to U$ with

$$c_0(0) = c_1(0), \qquad c_0(1) = c_1(1)$$

are called *homotopic [in U]* if there exists a *homotopy* between them, that is, a continuous map

$$\begin{aligned} H : [0,1] \times [0,1] &\to U \\ (t,\tau) &\mapsto H(t,\tau) \end{aligned}$$

such that

$$H(\cdot,0) = c_0, \qquad H(\cdot,1) = c_1$$

and

$$H(0,\tau) = c_0(0) = c_1(0), \qquad H(1,\tau) = c_0(1) = c_1(1).$$

A closed curve $c : [0,1] \to U$ is called *null-homotopic [in U]* if it is homotopic to the constant curve $c_1(t) = c(0)$.

Definition 18.2 (concatenation of curves). For continuous curves $c_1, c_2, c : [0,1] \to U$ with

$$c_1(1) = c_2(0),$$

define the *concatenation* $c_1c_2 : [0,1] \to U$ by

$$c_1c_2(t) = \begin{cases} c_1(2t) & \text{for } 0 \le t \le \frac{1}{2} \\ c_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

and the backward traced curve $c^{inv} : [0,1] \to U$ by

$$c^{inv}(t) = c(1-t).$$

Lemma 18.3. *Let $c : [0,1] \to U$ be a continuous curve, and let $\phi : [0,1] \to [0,1]$ be a continuous map with*

$$\phi(0) = 0, \qquad \phi(1) = 1.$$

Then

[1] This section is essentially copied verbatim from Dirk Ferus' notes, where you can also find the proofs.

(i) cc^{inv} *is null-homotopic.*
(ii) $c \circ \phi$ *is homotopic to* c.

Lemma 18.4. *Let* $c_1, c_2, c_3 : [0,1] \to U$ *be continuous curves with*

$$c_1(1) = c_2(0), \qquad c_2(1) = c_3(0).$$

Then $(c_1 c_2)c_3$ *and* $c_1(c_2 c_3)$ *are homotopic in* U.

Theorem and Definition 18.5 (fundamental group)**.** *Let* $U \in \mathbb{C}$ *be an open subset, and let* $z_0 \in U$. *In the set of all closed curves* $c : [0,1] \to U$ *with* $c(0) = c(1) = z_0$ *(the set of* loops at z_0*), homotopy defines an equivalence relation. The concatenation of curves defines a multiplication on the set* $\pi_1(U, z_0)$ *of equivalence classes, which turns this set into a group. It is called the* fundamental group of U with base point z_0. *The neutral element of the fundamental group is the class of the constant curve* $t \mapsto z_0$, *and the inverse element of the class a curve* c *is the class of* c^{inf}. *The class of a curve* c *is denoted by* $[c] \in \pi_1(U, z_0)$. *So we have*

$$[c_1][c_2] = [c_1 c_2], \qquad [c]^{-1} = [c^{inv}].$$

- The fundamental group $\pi_1(U, z_0)$ depends on the base point. But if U is a domain (or any other path-connected topological space) and $z_1 \in U$, then there is a curve $\gamma : [0,1] \to U$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. The map

  $$\pi_1(U, z_0) \ni [c] \longmapsto [\gamma c \gamma^{inv}] \in \pi_1(U, z_1)$$

  defines an group isomorphism $\pi_1(U, z_0) \longrightarrow \pi_1(U, z_1)$.

Theorem and Definition 18.6 (simply connected)**.** *Let* U *be a domain in* $\mathbb{C}$ *(or a path-connected* 19
topological space), and $U \neq \emptyset$. *Then the following statements are equivalent:*

(i) *Every closed curve* $c : [0,1] \to U$ *is null-homotopic in* U.
(ii) $\pi_1(U, z_0) = \{1\}$ *for all* $z_0 \in U$.
(iii) $\pi_1(U, z_0) = \{1\}$ *for at least one* $z_0 \in U$.
(iv) *Any two curves* $c_0, c_1 : [0,1] \to U$ *with* $c_0(0) = c_1(0)$, $c_0(1) = c_1(1)$ *are homotopic in* U.

If one and hence all of these statements are true, then U *is called* simply connected.

Examples 18.7. • Convex sets are simply connected.
- Star-shaped sets are simply connected.
- The set $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is not convex, but star-shaped.
- The punctured plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is not simply connected, and one can show that $\pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$.
- The fundamental group of the twice punctured plane $\mathbb{C} \setminus \{0,1\}$ is not abelian. It is isomorphic to the free group with two generators.

19 The monodromy theorem

Theorem 19.1 (monodromy theorem)**.** *Let* $G \in \mathbb{C}$ *be a domain and* $z_0 \in G$. *Suppose a function element* (f, U) *with* $z_0 \in U$ *can be continued analytically along any curve* $c : [0,1] \to U$ *beginning in* z_0. *If* c_0 *and* c_1 *are two homotopic curves in* G *beginning in* z_0 *and ending in* z_1, *and if* $(f_{0,t}, U_{0,t})_t$ *and* $(f_{1,t}, U_{1,t})_t$ *are analytic continuations of* (f, U) *along* c_0 *and* c_1, *respectively, then*

$$(f_{0,1}, U_{0,1}) \overset{z_1}{\sim} (f_{1,1}, U_{1,1}).$$

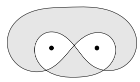
20 Homology of curves[2] ⑳

Let $U \subseteq \mathbb{C}$ be an open subset.

- A 0-*chain in* U is a formal sum
$$a_1 z_1 \oplus a_2 z_2 \oplus \ldots \oplus a_n z_n$$
of points $z_k \in U$ with coefficients $a_k \in \mathbb{Z}$.
- The *abelian group of* 0-*chains in* U is denoted by $C_0(U)$.
- A 1-*chain in* U is a formal sum
$$a_1 c_1 \oplus a_2 c_2 \oplus \ldots \oplus a_n c_n \tag{6}$$
of curves $c_k : [0,1] \to U$ with coefficients $a_k \in \mathbb{Z}$.
- The *abelian group of* 1-*chains in* U is denoted by $C_1(U)$.
- For a curve $c : [0,1] \to U$, let
$$\partial c = c(1) \ominus c(0) \in C_0(U).$$
For a 1-chain (6), let
$$\begin{aligned}\partial\Big(\bigoplus_{k=1}^n a_k c_k\Big) &= \bigoplus_{k=1}^n a_k\,\partial c_k \\ &= \Big(\bigoplus_{k=1}^n a_k c_k(1)\Big) \ominus \Big(\bigoplus_{k=1}^n a_k c_k(0)\Big).\end{aligned}$$
This defines a group homomorphism $\partial : C_1 \to C_0$ called the *boundary operator*.
- A 1-chain is called *closed* or a 2-*cycle* if $\partial c = 0$.
- The 1-chain $c = \bigoplus_k a_k c_k$ in U is closed if and only if for all $p \in U$
$$\sum_{k:c_k(1)=p} a_k = \sum_{k:c_k(0)=p} a_k$$
- Let f be a holomorphic function on U. For a 1-chain
$$c = \bigoplus_{k=1}^n a_k c_k$$
in U define
$$\int_c f(z)\,dz = \sum_{k=1}^n \int_{c_k} f(z)\,dz.$$
For a 0-chain $\bigoplus_{k=1}^n b_k z_k$, define
$$\int_{\bigoplus_{k=1}^n b_k z_k} f = \sum_{k=1}^n a_k f(z_k).$$
Then
$$\int_c f'(z)\,dz = \int_{\partial c} f.$$

[2]There are different homology theories. The one we consider here is called "singular homology with integer coefficients".

- Let $\Delta = \{(t_1, t_2) \in \mathbb{R}^2 \,|\, t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1\}$. A *singular triangle in* U is a continuous map
$$\tau : \Delta \to U.$$
A 2-*chain in* U is a formal sum
$$a_1 \tau_1 \oplus \ldots \oplus a_n \tau_n$$
of singular triangles τ_k with coefficients $a_k \in \mathbb{Z}$.
- The *abelian group of* 2-*chains in* U is denoted by $C_2(U)$.
- For a singular triangle $\tau : \Delta \to U$, let
$$\partial \tau = c_1 \ominus c_2 \oplus c_3 \in C_1(U),$$
where $c_k : [0,1] \to U$,
$$c_1(t) = \tau(t, 0), \qquad c_2(t) = \tau(0, t), \qquad c_3(t) = \tau(1-t, t).$$
For a 2-chain
$$\tau = \bigoplus_{k=1}^{n} a_k \tau_k,$$
let
$$\partial \tau = \bigoplus_{k=1}^{n} a_k \, \partial \tau_k.$$
This defines a group homomorphism $\partial : C_2(U) \to C_1(U)$, called the *boundary operator*.
- A 1-chain c is called 0-*homologous* if there is a 2-chain τ with $\partial \tau = c$.
- Two 1-chains c_1 and c_2 are called *homologous*, if $c_1 \ominus c_2$ is 0-homologous, that is, if there is a 2-chain τ with $c_1 = c_2 \oplus \tau$.
- A 0-homologous 1-chain is closed. The converse is not generally true: A closed 1-chain may not be 0-homologous.
- A constant curve $c(t) = z_0 \in U$ is 0-homologous. [exercise]
- $c \oplus c^{inv}$ is 0-homologous. [exercise]
- The concatenation $c_1 c_2$ of two curves with $c_0(1) = c_1(0)$ is homologous to $c_1 \oplus c_2$.
- If c_1 and c_2 are loops at z_0 in U then the concatenation $c_1 c_2 c_1^{inv} c_2^{inv}$ is null-homologous in U. [exercise]
- If c_1 and c_2 are homotopic in U then c_1 and c_2 are homologous in U. [exercise]
- The converse is not generally true: A 0-homologous curve may not be null-homotopic.



- If the closed curves c_1 and c_2 are freely homotopic in U, then c_1 and c_2 are homologous in U. [exercise]

This is the final version of Cauchy's integral theorem: (21)

Theorem 20.1 (Cauchy's integral theorem, homology version). *If f is holomorphic on U and the curve $c : I \to U$ is 0-homologous in U, then*
$$\int_c f(z) \, dz = 0.$$

21 Winding number

Definition 21.1. The *support* a 1-chain $c = \bigoplus_k a_k c_k$ is
$$|c| = \bigcup_{k : a_k \neq 0} c_k([0,1]),$$

and the *support* a 2-chain $\tau = \bigoplus_k a_k \tau_k$ is

$$|\tau| = \bigcup_{k:a_k \neq 0} \tau_k(\Delta).$$

Definition 21.2 (winding number)**.** The *winding number* of a 1-cycle c around a point $z_0 \notin |c|$, also called the *index* of z_0 with respect to c, is defined as

$$\operatorname{ind}_c(z_0) = \frac{1}{2\pi i} \int_c \frac{dz}{z - z_0}.$$

Example 21.3. For $c(t) = a + re^{2\pi i n t}$ on $[0, 1]$,

$$\operatorname{ind}_c(a) = n \qquad \text{and} \qquad \operatorname{ind}_c(z) = 0 \quad \text{if} \quad |z_0 - a| > r.$$

Theorem 21.4. *(i) The winding number* $\operatorname{ind}_c(z_0)$ *of* 1*-cycle is an integer.*

(ii) If c is a 1*-cycle and z_0 and z_1 are contained in the same connected component of* $\mathbb{C} \setminus |c|$*, then*

$$\operatorname{ind}_c(z_0) = \operatorname{ind}_c(z_1).$$

(iii) If two 1*-cycles c and $\tilde{c}$ are homologous in* $\mathbb{C} \setminus \{z_0\}$*, then*

$$\operatorname{ind}_c(z_0) = \operatorname{ind}_{\tilde{c}}(z_0).$$

- crossing rule for winding numbers

Theorem 21.5 (Artin's homology criterion)**.** *A* 1*-cycle c is* 0*-homologous in an open subset $U \subseteq \mathbb{C}$ if* (22)
and only if $\operatorname{ind}_c a = 0$ *for every* $a \in \mathbb{C} \setminus U$.

A proof can be found in Dirk Ferus' lecture notes.

Corollary 21.6 (converse of Cauchy's integral theorem)**.** *If c is not* 0*-homologous in U, then there is a holomorphic function f on U such that*

$$\int_c f(z)\, dz \neq 0.$$

Definition 21.7 (simply bounding cycle)**.** Let $B \in \mathbb{C}$ be a compact subset. A 1-cycle *bounds B simply* if

(i) $|c| \subseteq \partial B$,
(ii) $\operatorname{ind}_c z = 1$ if z is an interior point of B,
(iii) $\operatorname{ind}_c z = 0$ if $z \in \mathbb{C} \setminus B$.

Examples 21.8. (i)–(iv) disk, rectangle, annulus, Jordan domain

(v) If the 1-cycle c bounds the compact set B simply, and if $K_1, \ldots, K_n$ are disjoint closed disks in the interior B° of B, then

$$B \setminus \bigcup_{j=1}^{n} K_j^\circ$$

is simply bounded by

$$c \ominus \partial K_1 \ominus \ldots \ominus \partial K_n.$$

Theorem 21.9 (winding number version of Cauchy's integral formula). *Let f be holomorphic on U and c be a 0-homologous 1-cycle in U. Then for every $z_0 \in U \setminus |c|$,*

$$\operatorname{ind}_c(z_0)\, f(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0}\, dz.$$

In particular, if c bounds the compact set $B \subseteq U$ simply, and $z_0 \in B^\circ$, then

$$f(z_0) = \frac{1}{2\pi} \int_c \frac{f(z)}{z - z_0}\, dz.$$

22 The residue theorem

Definition 22.1 (residue). If f has an isolated singularity at z_0, then f is holomorphic on

$$A = \{z \mid 0 < |z - z_0| < R\}$$

for some $R > 0$, and f can be expanded into a Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ on A. The *residue of f at z_0* is

$$\operatorname{Res}(f, z_0) = a_{-1}.$$

- By the Laurent series theorem, if $0 < r < R$,

$$Res(f, z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} f(z)\, dz.$$

Theorem 22.2 (residue theorem v1). *Let f be holomorphic in a domain U except for a set S of isolated singularities. If a 1-cycle bounds a compact set $A \subseteq U$ simply, and $S \cap \partial B = \emptyset$, then*

$$\frac{1}{2\pi i} \int_c f(z)\, dz = \sum_{z \in S \cap B} \operatorname{Res}(f, z).$$

Theorem 22.3 (residue theorem v2). *If f is holomorphic on a domain U except for a set S of isolated singularities, and if c is a cycle in $U \setminus S$ that is 0-homologous in U, then the set*

$$\{z \in S \mid \operatorname{ind}_c(z) \neq 0\}$$

is finite and

$$\frac{1}{2\pi i} \int_c f(z)\, dz = \sum_{z \in S} \operatorname{ind}_c(z) \operatorname{Res}(f, z).$$

Example 22.4. Iff and g are holomorphic in a neighborhood of z_0 and g has a simple zero at z_0, (23)
then

$$\operatorname{Res}(\frac{f}{g}, z_0) = \frac{f(z_0)}{g'(z_0)}.$$

Example 22.5. If f has a zero of order m or a pole of order $-m$ at z_0, then

$$\operatorname{Res}\Big(\frac{f'}{f}, z_0\Big) = m.$$

Theorem 22.6 (counting zeros and poles with an integral, and with a winding number)**.** *Let f be meromorphic on U and let $B \subseteq U$ be a compact region simply bounded by the 1-cycle c. Assume that f has no poles or zeros on the boundary of B. Let $p_1, \ldots, p_m$ be the poles of f in B, and let $a_1, \ldots, a_m \in \mathbb{Z}_{>0}$ be their pole orders. Let $q_1, \ldots, q_n$ be the zeros of f in B, and let $b_1, \ldots, b_n \in \mathbb{Z}_{>0}$ be their orders. Then*

$$\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{n} b_k - \sum_{k=1}^{m} a_k = \operatorname{ind}_{f\circ c}(0).$$

- Used in the proof of Theorem 22.6:

Lemma 22.7 (substitution rule for integrals along continuous curves)**.** *If $c : [a, b] \to \mathbb{C}$ is a continuous curve, ϕ is holomorphic in a neighborhood of $|c|$, and f is holomorphic in a neighborhood of $|\phi \circ c| = \phi(|c|)$, then*

$$\int_{\phi\circ c} f(z)\, dz = \int_c (f \circ \phi)(z)\, \phi'(z)\, dz.$$

Theorem 22.8 (Rouché)**.** *Let f and g be holomorphic on U, and let $B \subseteq U$ be a compact region simply bounded by a continuous curve c. Suppose*

$$|f(z) - g(z)| < |f(z)| \quad \text{for all} \quad z \in \partial B.$$

Then f and g have the same number of zeros in B, counted with multiplicities (i.e., order).

Example 22.9. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{e}$