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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

Exercise Sheet 2

Due before the lecture on Thursday, May 4, 2017.

Exercise 5: Radius of convergence. (4 pts)

Let $a \in \mathbb{C} \setminus \{0\}$. Determine the radius of convergence of the following power series:

$$\text{(i)} \sum_{k=0}^{\infty} a^k z^k \qquad \text{(ii)} \sum_{k=0}^{\infty} a^{k^2} z^k \qquad \text{(iii)} \sum_{k=0}^{\infty} z^{k!} \qquad \text{(iv)} \sum_{k=0}^{\infty} k! z^k \,.$$

Exercise 6: The Cauchy–Riemann equations. (4 pts)

Let U be a domain in $\mathbb{C}$, and $f : U \to \mathbb{C}$ a holomorphic function.

1. Show that if the real part or the imaginary part of f is constant, then f is constant.
2. Show that if the argument or the absolute value of f is constant, then f is constant.[1]

Exercise 7: Differentiability and the Cauchy–Riemann equations. (4 pts)

1. Check whether the function $f : \mathbb{C} \to \mathbb{C}$ given by
$$f(x + iy) = -4xy + 2i(x^2 - y^2 + 3)$$
satisfies the Cauchy–Riemann equations. Conclude whether f is holomorphic, and express f in terms of z and $\bar{z}$ only.
2. Consider the function $g : \mathbb{C} \to \mathbb{C}$ given by
$$g(x + iy) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$
Show that $\lim_{z \to 0} \frac{g(z)-g(0)}{z}$ is zero when $z \to 0$ along any straight line, and equals $\frac{1}{2}$ when $z \to 0$ along the curve $x = y^2$.
Conclude that at $z = 0$, the function g satisfies the Cauchy–Riemann equations, but is not differentiable, neither complex nor real.

Exercise 8: Harmonic functions. (4 pts)

1. Which of the following functions are harmonic?
$$\text{(i) } u(x, y) = e^x \qquad \text{(ii) } v(x, y) = e^x(x \cos y - y \sin y)$$
Find the harmonic conjugate where applicable.
2. Find the most general form for which the polynomial
$$ax^2 + 2bxy + cy^2$$
is the real part of a holomorphic function h. Construct this holomorphic function, and express it in terms of z.

[1] This implies that if, for instance, the arguments of two holomorphic functions are equal, they only differ by a positive multiplicative constant.

Bonus.

What does the following sequence of plots depict?