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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

Exercise Sheet 3

Due before the lecture on Thursday, May 11, 2017.

Exercise 9: Special Möbius transformations. (4 pts)

There are six permutations σ of the triple $(0, 1, \infty)$. Each of them yields a Möbius transformation f_σ determined by $f_\sigma(p) = \sigma(p)$ for $p \in (0, 1, \infty)$. Find these.

Exercise 10: Fixed points of Möbius transformations, part I. (4 pts)

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$, $ad - bc = 1$, that is not the identity. Show that the fixed points of f are

$$z_\pm = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

Calculate $f'(z_\pm)$ and deduce that $f'(z_+)\, f'(z_-) = 1$.

Exercise 11: Fixed points of Möbius transformations, part II. (4 pts)

1. Let g_1 be a Möbius transformation having ∞ as only fixed point, and g_2 a Möbius transformation with fixed points 0 and ∞. Show that there exist $r \in \mathbb{C} \setminus \{0\}$ and $s \in \mathbb{C} \setminus \{0, 1\}$ such that $g_1(z) = z + r$, $g_2(z) = sz$ for all z.
2. Let f_1 be a Möbius transformation with only one fixed point in $\hat{\mathbb{C}}$, and let f_2 be a Möbius transformation with two different fixed points in $\hat{\mathbb{C}}$. Show that there exist $r \in \mathbb{C} \setminus \{0\}$, $s \in \mathbb{C} \setminus \{0, 1\}$ and Möbius transformations τ_1, τ_2 such that

$$\tau_1 \circ f_1 \circ \tau_1^{-1}(z) = z + r \qquad \text{and} \qquad \tau_2 \circ f_2 \circ \tau_2^{-1}(z) = sz \qquad \text{for all } z \in \hat{\mathbb{C}}.$$

Exercise 12: Stereographic projection. (4 pts)

Stereographic projection from the North Pole $s : \mathbf{S}^2 \to \hat{\mathbb{C}}$ is given by

$$(x_1,\, x_2,\, x_3)^T \mapsto \frac{x_1 + ix_2}{1 - x_3}.$$

Its inverse $s^{-1} : \hat{\mathbb{C}} \to \mathbf{S}^2$ is given by

$$x + iy \mapsto \frac{1}{1 + x^2 + y^2} \left(2x,\, 2y,\, x^2 + y^2 - 1\right)^T \qquad \text{for } x + iy \in \mathbb{C},$$
$$\infty \mapsto (0,\, 0,\, 1)^T.$$

Every $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ can be uniquely identified with a map $\tilde{f} : \mathbf{S}^2 \to \mathbf{S}^2$, given by $\tilde{f} = s^{-1} \circ f \circ s$. Determine $\tilde{f}_i$ for

(i) $f_1(z) = \bar{z}$ (ii) $f_2(z) = \frac{1}{\bar{z}}$ (iii) $f_3(z) = -z$ (iv) $f_4(z) = z^2$

and describe the action of $\tilde{f}_i$ on $\mathbf{S}^2$. [1]

Bonus.

Visualize the sequence f_n, where

$$f_n(z) = \left(1 + \frac{z}{n}\right)^n.$$

What is its limit? As in Exercise 12, also describe the induced maps $\tilde{f}_n : \mathbf{S}^2 \to \mathbf{S}^2$.

[1] Hint: For (iv), you may want to use *cylindrical coordinates* $(x_1,\, x_2,\, x_3) = (r\cos\varphi,\, r\sin\varphi,\, x_3)$.