Technische Universität Berlin
Institut für Mathematik

Prof. Dr. Boris Springborn
Lara Skuppin

Berlin Mathematical School

Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

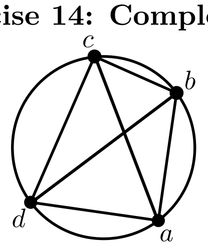
Exercise Sheet 4

Due before the lecture on Thursday, May 18, 2017.

Exercise 13: Permutations and the complex cross-ratio. (4 pts)

1. Show that if $\sigma \in S_4$ is the identity or a product of two transpositions of two disjoint pairs, that is, $(1)(2)(3)(4)$, $(12)(34)$, $(13)(24)$, or $(14)(23)$, then $\operatorname{cr}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}) = \operatorname{cr}(z_1, z_2, z_3, z_4)$.
2. Express $\operatorname{cr}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ in terms of $q := \operatorname{cr}(z_1, z_2, z_3, z_4)$, for $\sigma \in S_4$.[1]

Exercise 14: Complex cross-ratio and Ptolemy's theorem. (4 pts)



Let a, b, c, d be four points on a circle in cyclic order.

1. Show that $\angle acb = \angle adb$ by consideration of the cross-ratio.
2. From Exercise 13, recall that $\operatorname{cr}(z_1, z_2, z_3, z_4) + \operatorname{cr}(z_1, z_3, z_2, z_4) = 1$ for any four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$. Deduce *Ptolemy's theorem*:

$$|a-c||b-d| = |a-b||c-d| + |b-c||d-a|.$$

Exercise 15: Line integrals. (4 pts)

Let $U \subseteq \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be holomorphic. Show that for any closed curve γ in U, the integral

$$\int_\gamma \overline{f(z)} f'(z) dz$$

is purely imaginary.

Exercise 16: A sufficient condition for injectivity. (4 pts)

Let $U \subseteq \mathbb{C}$ be open and convex and let $f : U \to \mathbb{C}$ be holomorphic [with continuous derivative[2]] and assume

$$|f'(z) - 1| < 1 \text{ for all } z \in U.$$

Show that f is injective on U.
Hint: For all $z_1, z_2 \in U$, consider the integral of $f'(z) - 1$ along the segment $[z_1, z_2]$, and estimate its absolute value.

Bonus.

Check out the calendar at http://www.mathcalendar.net and pick your favorite portrait of the month.[3] What do you find interesting about it?

[1]Hint: Apply a suitable Möbius transformation and recall from Exercise 9 that the six Möbius transformations corresponding to the permutations of $\{0, 1, \infty\}$ are the following:

$$f_{(0)(1)(\infty)}(z) = z \qquad f_{(01)(\infty)}(z) = 1 - z \qquad f_{(0\infty)(1)}(z) = \frac{1}{z}$$

$$f_{(0)(1\infty)}(z) = \frac{z}{z-1} \qquad f_{(01\infty)}(z) = \frac{1}{1-z} \qquad f_{(\infty 10)}(z) = 1 - \frac{1}{z}.$$

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[2]We will later see that this condition is always satisfied.
[3]With special thanks to Carl Lutz for the suggestion.