TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. Boris Springborn
Lara Skuppin

Berlin Mathematical School

COMPLEX ANALYSIS I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

EXERCISE SHEET 6

Due before the lecture on Thursday, June 01, 2017.

Exercise 21: Contour integrals and C^1-homotopies. (4 pts)

Let $U \subseteq \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be a holomorphic function.

1. Consider continuously differentiable curves $\alpha_0, \alpha_1 : [a, b] \longrightarrow U$ with common startpoint p and common endpoint q, that is, $\alpha_0(a) = p = \alpha_1(a)$ and $\alpha_0(b) = q = \alpha_1(b)$. The curves α_0 and α_1 are said to be C^1-*homotopic* if there is a C^1-map

$$H : [0, 1] \times [a, b] \longrightarrow U$$

with $H(0, t) = \alpha_0(t)$, $H(1, t) = \alpha_1(t)$ for all t in $[a, b]$, and $H(s, a) = p$, $H(s, b) = q$ for all s in $[0, 1]$.
Show that if α_0 and α_1 are C^1-homotopic, then

$$\int_{\alpha_0} f(z)\, dz = \int_{\alpha_1} f(z)\, dz\,.$$

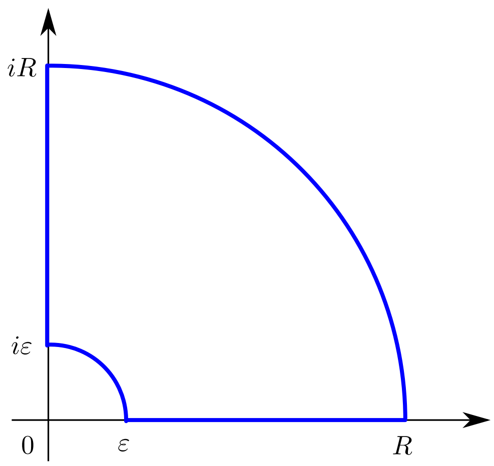
2. Consider continuously differentiable *closed* curves $\beta_0, \beta_1 : [a, b] \longrightarrow U$. The curves β_0 and β_1 are said to be *freely* C^1-*homotopic* if there exists a C^1-map

$$H : [0, 1] \times [a, b] \longrightarrow U$$

with $H(0, t) = \beta_0(t)$, $H(1, t) = \beta_1(t)$ for all t in $[a, b]$, and $H(s, a) = H(s, b)$ for all s in $[0, 1]$.
Show that if β_0 and β_1 are freely C^1-homotopic, then

$$\int_{\beta_0} f(z)\, dz = \int_{\beta_1} f(z)\, dz\,.$$

Exercise 22: Real integrals from Cauchy's theorem. (4 pts)



Integrate the function $f(z) = \dfrac{e^{iz}}{z}$ along the cycle shown in the figure. Apply Cauchy's integral theorem to show

$$\int_0^\infty \frac{\cos(x) - e^{-x}}{x}\, dx = 0\,, \qquad \int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$