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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

Exercise Sheet 7

Due before the lecture on Thursday, June 08, 2017.

Exercise 23: Cauchy's integral formula I. (4 pts)

1. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Let $z_0 \neq z_1$ be complex numbers and R a real number such that $R > |z_0|$ and $R > |z_1|$. Show that

$$\int_{|z|=R} \frac{f(z)}{(z-z_0)(z-z_1)}\,dz = 2\pi i\,\frac{f(z_1)-f(z_0)}{z_1-z_0}\,. \qquad (*)$$

2. Recall the complex cosine function $\cos(z) = \frac{1}{2}(e^{iz}+e^{-iz})$. For k in $\mathbb{Z}$, determine

$$\int_{|z|=9} \frac{\cos(z)}{z^k}\,dz\,.$$

Exercise 24: Cauchy's integral formula II. (4 pts)

Let p be a polynomial of degree n and $R > 0$ such that $|z_0| < R$ for all zeros z_0 of p. Show that

$$\int_{|z|=R} \frac{p'(z)}{p(z)} = 2\pi i n.$$

Exercise 25: Power series theorem. (4 pts)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and let $C > 0$, $n \in \mathbb{N}$ and $R > 0$ such that

$$|f(z)| \leq C|z|^n$$

for all z in $\mathbb{C}$ with $|z| \geq R$. Prove that f is a polynomial of degree at most n.

Exercise 26: Liouville's theorem. (4 pts)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic such that $\mathbb{C} \setminus f(\mathbb{C})$ contains a non-empty open disc. Show that f is constant.
This means that the values of an entire function are dense in $\mathbb{C}$.

Bonus. (2 pts)

Prove Liouville's theorem using $(*)$: Given a bounded entire function f, estimate the absolute value of the integral on the left side in terms of R, and let $R \to \infty$. Deduce that f is constant.