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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

Exercise Sheet 8

Due before the lecture on Thursday, June 15, 2017.

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk.

Exercise 27: Identity theorem. (4 pts)

Show that there is no entire function $f : \mathbb{C} \to \mathbb{C}$ with

$$f\left(\frac{1}{n}\right) = \frac{n}{2n-1}$$

for all $n \in \mathbb{N}$.

Exercise 28: Maximum principle. (4 pts)

Let f be holomorphic on a domain U, and let $K \subset U$ be compact with non-empty interior. Prove that if $|f|$ is constant on ∂K, then f has a zero in $\operatorname{int}(K)$ or is constant.

Exercise 29: Schwarz' lemma I. (4 pts)

Let $f : \mathbb{D} \to U \subset \mathbb{C}$ be biholomorphic. Show that

$$|f'(0)| \geq \operatorname{dist}(f(0), \partial U).$$

Remark: $\operatorname{dist}(p, \partial U) := \sup\{R > 0 \mid B_R(p) \subset U\}$, where $B_R(p)$ is a disk of radius R around $p \in U$.

Exercise 30: Schwarz' lemma II. (4 pts)

For $a \in \mathbb{D}$, define the Möbius transformation T_a by

$$T_a(z) := \frac{z-a}{1-\bar{a}z}.$$

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $z_0 \in \mathbb{D}$. Apply the Schwarz lemma to the function $F = T_{f(z_0)} \circ f \circ T_{z_0}^{-1}$ and show that

$$\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \leq \left|\frac{z-z_0}{1-\overline{z_0}z}\right|$$

for all $z \in \mathbb{D}$. Conclude that the following estimate[1] holds for all $z \in \mathbb{D}$:

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}.$$

[1] Note to the experts: This implies that f is *contracting* with respect to the hyperbolic metric on the Poincaré disk $\mathbb{D}$, see `https://en.wikipedia.org/wiki/Poincar%C3%A9_disk_model` .