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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

Exercise Sheet 9

Due before the lecture on Thursday, June 22, 2017.

Exercise 31: Singularities. (4 pts)

Determine and classify the singularities of the following functions:

(i) $f_1(z) = \dfrac{z^2-1}{z^3-1}$ (ii) $f_2(z) = \dfrac{z^2+iz}{(z^2+1)^3}\cosh(\pi z/2)$ (iii) $f_3(z) = \dfrac{z}{e^z-1}$ (iv) $f_4(z) = z\sin\left(z+\dfrac{1}{z}\right)$

Exercise 32: Reflection principle. (4 pts)

Let f be an entire function with $f(\mathbb{R}) \subseteq \mathbb{R}$ and $f(i\mathbb{R}) \subseteq i\mathbb{R}$. Show that f is an odd function, that is, $f(-z) = -f(z)$ for all z in $\mathbb{C}$.

Exercise 33: Singularities at infinity. (4 pts)

Let U be ~~an open subset of~~ a domain in $\mathbb{C}$ such that $\mathbb{C} \setminus U$ is compact. Let $f : U \to \mathbb{C}$ be holomorphic. Then $z_0 = \infty$ is called a removable singularity/pole of order m/essential singularity of f, if the function $g(z) = f(\frac{1}{z})$ has a removable singularity/pole of order m/essential singularity in $z = 0$.

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that f has a removable singularity in $z = \infty$ if and only if f is constant.
2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that f has a pole of order m in $z = \infty$ if and only if f is a polynomial of order m.
3. Let $f, g : U \to \mathbb{C}$ be holomorphic with removable singularities in $z = \infty$. Show that $f = g$ if $f(k) = g(k)$ for all $k \in U \cap \mathbb{Z}$.

Exercise 34: Biholomorphisms. (4 pts)

Let $0 < r < 1$. Show: $\mathbb{D} \setminus \{0\}$ and $\mathbb{D} \setminus \{z \in \mathbb{D} \mid |z| \leq r\}$ are diffeomorphic, but there is no biholomorphic map between them.