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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

Exercise Sheet 11

Due before the lecture on Thursday, July 6, 2017.

The numbering of definitions, propositions, etc. refers to the 'growing summary of contents' as linked on the course homepage.

Exercise 38: Lebesgue number. (4 pts)

Prove the Lebesgue lemma:

> Let $(U_j)_{j\in J}$ be an open cover of a compact metric space X. Then there is an $\epsilon > 0$, called a *Lebesgue number* of the cover, such that for every subset $A \subseteq X$ with $\operatorname{diam}(A) < \epsilon$, there is a $j \in J$ with $A \subseteq U_j$.

Hint: One possibility is to consider $\rho : X \to \mathbb{R}$ defined by

$$\rho(x) := \sup\{r \in \mathbb{R} \mid \text{there exists } j \in J \text{ with } B_r(x) \subseteq U_j\},$$

where $B_r(x)$ denotes the open ball $\{y \in X \mid d(x,y) < r\}$. Show that ρ is continuous and deduce the Lebesgue lemma from that.

Exercise 39: Analytic continuation. (4 pts)

Prove Lemma 17.11:

> Let (f_t, U_t) be an analytic continuation along a curve $c : [0,1] \to \mathbb{C}$. Then there is a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ and a sequence of function elements $(g_k, D_k)_{1\le k\le n-1}$, where D_k are disks, such that
>
> $$(f_t, U_t) \overset{c(t)}{\sim} (g_k, D_k) \quad \text{for all} \quad t \in [t_{k-1}, t_{k+1}].$$

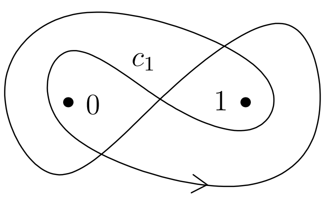
Exercise 40: Contour integration. (4 pts)

Let $f : U \to \mathbb{C}$ be holomorphic and $\gamma : [a,b] \to U$ be a curve that is piecewise C^1. Show that the integral

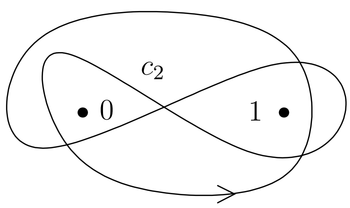
$$\int_\gamma f(z)\,dz$$

defined for continuous curves (as in Definition 17.13) has the same value as the integral as defined for piecewise C^1-curves (in Definition 7.1).

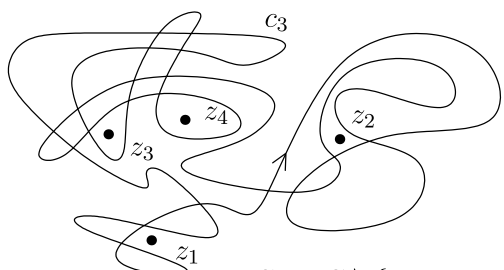
Exercise 41: Homotopy. (4 pts)



$G_1 = \mathbb{C} \setminus \{0,1\}$



$G_2 = \mathbb{C} \setminus \{0,1\}$



$G_3 = \mathbb{C} \setminus \{z_1, z_2, z_3, z_4\}$

Which of the above curves c_i are null-homotopic in the given region G_i?