Technische Universität Berlin
Institut für Mathematik

Prof. Dr. Boris Springborn
Lara Skuppin

Berlin Mathematical School

Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/ComplexAnalysis/

Exercise Sheet 12

Due before the lecture on Thursday, July 13, 2017.

Exercise 42: Homology. (4 pts)

Let U be an open subset of $\mathbb{C}$. Let $c, c_1, c_2 : [0,1] \to U$ be curves. Show directly, i.e., without using Exercise 43:

1. If c is constant, then the c is null-homologous.
2. The concatenation cc^{inv} is null-homologous.
3. If c_1 and c_2 are loops at z_0 in U, then $c_1 c_2 c_1^{inv} c_2^{inv}$ is null-homologous.

Exercise 43: Homotopy and homology. (4 pts)

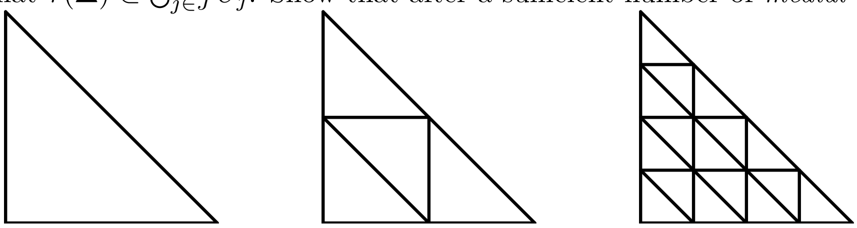
Let U be an open subset of $\mathbb{C}$, and let $c_0, c_1 : [0,1] \to U$ be curves.

1. Assume $c_1(0) = c_2(0) =: p$ and $c_1(1) = c_2(1) =: q$ and suppose that c_1 and c_2 are homotopic. Show that then c_1 and c_2 are homologous.
2. Suppose c_1 and c_2 are *closed* curves, $c_j(1) = c_j(0)$, and suppose that c_1 and c_2 are *freely homotopic*. Show that c_1 and c_2 are homologous.

   Note: The curves c_1 and c_2 are freely homotopic if there is a continuous map $H : [0,1] \times [0,1] \to U$ satisfying
   (a) $H(0,\tau) = H(1,\tau)$ for all τ in $[0,1]$ (the curves $t \mapsto H(t,\tau)$ are closed), and
   (b) $H(t,0) = c_1(t)$ and $H(t,1) = c_2(t)$ for all t in $[0,1]$.

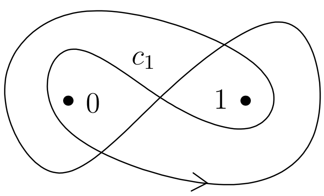
Exercise 44: Open covers and subdivision. (4 pts)

Let Δ be a closed triangle in $\mathbb{R}^2$, and let $\tau : \Delta \to \mathbb{C}$ be continuous. Suppose that $\{U_j\}_{j \in J}$ is a family of open sets such that $\tau(\Delta) \subset \bigcup_{j \in J} U_j$. Show that after a sufficient number of *medial subdivisions* of Δ,

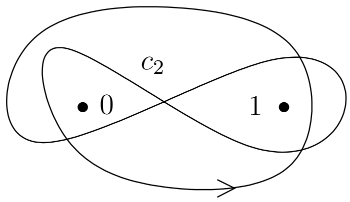


the image of every resulting small triangle is contained in some U_j.

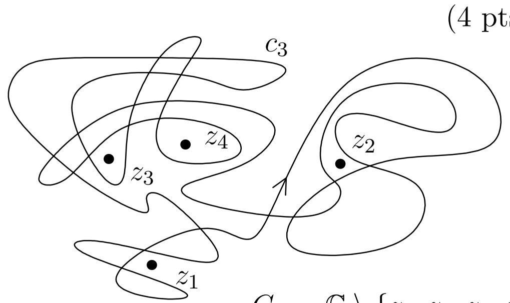
Exercise 45: Homology. (4 pts)



$G_1 = \mathbb{C} \setminus \{0,1\}$



$G_2 = \mathbb{C} \setminus \{0,1\}$



$G_3 = \mathbb{C} \setminus \{z_1, z_2, z_3, z_4\}$

1. Which of the above curves c_i are zero-homologous in the given region G_i?
2. Let $z_1, z_2, z_3 \in \mathbb{C}$ be three different points. Construct a zero-homologous curve in $\mathbb{C} \setminus \{z_1, z_2, z_3\}$ which is not null-homotopic in $\mathbb{C} \setminus \{z_i, z_j\}$ for all $i \neq j$.