1. Our task is to find a Möbius transformation f that maps the real axis to the unit circle and $C = \{z \mid |z - ih|^2 = r^2\}$ to a circle of radius $\tilde{r} < 1$ centered at 0. There are several approaches, below is just one possibility.

Recall that a Möbius transformation mapping $\mathbb{H}$ to $\mathbb{D}$ is given by $g(z) = \frac{z-i}{z+i}$. Since the setup in this exercise is symmetric with respect to the imaginary axis, our strategy will be to find a Möbius transformation h that preserves both the real and imaginary axes, that is, $h(z) = \lambda z$ for some $\lambda \neq 0$ in $\mathbb{R}$, such that $f = g \circ h$ will solve the problem.

Image of a circle under a Möbius transformation. The equation of $C = \{z \in \mathbb{C} \mid |z - z_0|^2 = R^2\}$ can be written as

$$z\bar{z} - \bar{z}_0 z - z_0 \bar{z} + z_0 \bar{z}_0 - R^2 = 0. \tag{1}$$

What is its image under the Möbius transformation T, when

$$T(z) = \frac{az+b}{cz+d}\ ? \tag{2}$$

The answer is

$$\begin{aligned} T(C) &= \{w \in \mathbb{C} \mid T^{-1}(w) \in C\} \\ &= \{w \in \mathbb{C} \mid |T^{-1}(w) - z_0|^2 = R^2\} \\ &= \{w \in \mathbb{C} \mid T^{-1}(w)\overline{T^{-1}(w)} - \bar{z}_0 T^{-1}(w) - z_0 \overline{T^{-1}(w)} + z_0\bar{z}_0 - R^2 = 0\}. \end{aligned} \tag{3}$$

The inverse of T is given by

$$T^{-1}(w) = \frac{dw - b}{-cw + a}. \tag{4}$$

Let us rearrange the equation in (3) as

$$\begin{aligned} T^{-1}(w)\overline{T^{-1}(w)} - \bar{z}_0 T^{-1}(w) - z_0\overline{T^{-1}(w)} + z_0\bar{z}_0 - R^2 &= 0 \\ \frac{dw-b}{-cw+a}\frac{\overline{dw-b}}{\overline{-cw+a}} - \bar{z}_0\frac{dw-b}{-cw+a} - z_0\frac{\overline{dw-b}}{\overline{-cw+a}} + z_0\bar{z}_0 - R^2 &= 0 \\ (dw-b)\overline{(dw-b)} - \bar{z}_0(dw-b)\overline{(-cw+a)} - z_0\overline{(dw-b)}(-cw+a) + & \\ (z_0\bar{z}_0 - R^2)\overline{(-cw+a)}(-cw+a) &= 0 \end{aligned}$$

and sort into coefficients of $w\bar{w}$, w and $\bar{w}$:

$$\begin{aligned} & w\bar{w}\left(d\bar{d} + \bar{z}_0\bar{c}d + z_0 c\bar{d} + c\bar{c}(z_0\bar{z}_0 - R^2)\right) \\ & -w\left(\bar{b}d + \bar{z}_0\bar{a}d + z_0\bar{b}c + (z_0\bar{z}_0 - R^2)\bar{a}c\right) \\ & -\bar{w}\left(b\bar{d} + \bar{z}_0 b\bar{c} + z_0 a\bar{d} + (z_0\bar{z}_0 - R^2)a\bar{c}\right) \\ & \ + \left(b\bar{b} + \bar{z}_0\bar{a}b + z_0 a\bar{b} + (z_0\bar{z}_0 - R^2)a\bar{a}\right) = 0 \end{aligned}$$

The midpoint of $T(C)$ is therefore

$$w_0 = \frac{b\bar{d} + \bar{z}_0 b\bar{c} + z_0 a\bar{d} + (z_0\bar{z}_0 - R^2)a\bar{c}}{d\bar{d} + \bar{z}_0\bar{c}d + z_0 c\bar{d} + c\bar{c}(z_0\bar{z}_0 - R^2)}. \tag{5}$$

In our ansatz for f, we have $a = c = \lambda, d = -b = i, z_0 = hi, R = r$. The condition that $f(C)$ is centered at 0 then becomes

$$\begin{aligned} 0 = b\bar{d} + \bar{z}_0 b\bar{c} + z_0 a\bar{d} + (z_0\bar{z}_0 - R^2)a\bar{c} &= -1 + (-ih)(-i)\lambda + ih\lambda(-i) + (h^2 - r^2)\lambda^2 \\ &= -1 + (h^2 - r^2)\lambda^2 \end{aligned}$$

Therefore we take $\lambda = \frac{1}{\sqrt{h^2-r^2}}$ as to also map the upper halfplane, and therefore C, to the interior, and not the exterior, of $\mathbb{D}$. Consequently

$$f(z) = g(\lambda z) = \frac{\frac{1}{\sqrt{h^2-r^2}}z - i}{\frac{1}{\sqrt{h^2-r^2}}z + i} = \frac{z - i\sqrt{h^2 - r^2}}{z + i\sqrt{h^2 - r^2}} \tag{6}$$

does the job as required, as well as any $\tilde{f} = e^{i\varphi} f$.

To get the radius of $f(C)$, calculate

$$\begin{aligned} \tilde{r} = |f(i(h + r))| = \frac{h + r - \sqrt{h^2 - r^2}}{h + r + \sqrt{h^2 - r^2}} &= \frac{\left(h + r - \sqrt{h^2 - r^2}\right)^2}{(h + r)^2 - (h^2 - r^2)} \\ &= \frac{h - \sqrt{h^2 - r^2}}{r}. \end{aligned} \tag{7}$$

2. Since $z \mapsto \ln(|z|)$ is harmonic on $\mathbb{C}\backslash\{0\}$ and zero on the unit circle, we take $\Psi(z) = C\ln(|z|)$, and determine $C = \frac{V}{\ln \tilde{r}}$. Note that Ψ is unique, due to the maximum principle for harmonic functions.

3. Take $\Phi = \Psi \circ f$. Then Φ is harmonic since f is holomorphic. Further, the boundary conditions match. Explicitly, this is

$$\Phi(z) = \frac{V}{\ln \frac{h-\sqrt{h^2-r^2}}{r}} \ln \left| \frac{z - i\sqrt{h^2 - r^2}}{z + i\sqrt{h^2 - r^2}} \right|. \tag{8}$$