1. Which of the two numbers is greater?
   (a) ◯ $\frac{1}{7}$ or ◯ 0.7
   (b) ◯ $\frac{3}{8}$ or ◯ 0.3

2. What is the area A of a circular disk with radius 1?

   $A =$ ________________

3. Fill out the following table.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(x)$					
$\cos(x)$					

4. Find all real solutions of the equation $(\sin(2x) - 1)(\cos(x) + 1) = 0$.

   $x \in$ ____

5. Write the results in the form $a + bi$ with $a, b \in \mathbb{R}$.

   (a) $(1 + i)(1 - i) =$ ________________________________

   (b) $\dfrac{1}{1 + i} =$ ________________________________

   (c) $i^{-1} =$ ________________________________

   (d) $e^{\frac{i\pi}{2}} =$ ________________________________

6. Calculate the absolute value and the argument.

   (a) $|1 + i| =$ ________________ (b) $\arg(1 + i) =$ ________________

7. Which functions are represented by the following power series?

   (a) $1 + x + x^2 + x^3 + x^4 + \ldots =$ ____________________

   (b) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \mp \ldots =$ ____________________

   (c) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \mp \ldots =$ ____________________

   (d) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots =$ ____________________

8. Calculate the derivative and an antiderivative of the function $f(x) = \cot(x)$.

   (a) $f'(x) =$ ____________________

   (b) $\int f(x)\,dx =$ ____________________

9. What are the Cauchy–Riemann equations?

   ____

10. Which of the following functions are harmonic?

    (a) $f_1(z) = \operatorname{Re}\sin(z)$
    (b) $f_2(z) = \operatorname{Arg}(z)$
    (c) $f_3(z) = (z+\bar{z})^3 - (z-\bar{z})^2$

11. Visualize the sequence

$$f_n(z) = \left(1 + \frac{z}{n}\right)^n .$$

12. What is a Möbius transformation?

13. What is the inverse of the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$, $ad - bc \neq 0$?

14. Find the Möbius transformation mapping $1 \mapsto 1$, $0 \mapsto i$, $-1 \mapsto -1$. What are the images of the four quadrants?

15. Does there exist a Möbius transformation mapping $\{\pm 1 \pm i\}$ to $\{-i, 0, i, 1\}$?

16. Prove or disprove: If $U \subseteq \mathbb{C}$ is a domain, $f : U \to \mathbb{C}$ is continuous and $\gamma : [t_0, t_1] \to \mathbb{C}$ is a C^1-curve, then

$$\left| \int_\gamma f(z) dz \right| \leq \text{length}(\gamma) \int_{t_0}^{t_1} |f(\gamma(t))| dt.$$

17. Compute

$$\int_{\partial R} z^2 + 5z + 2i\, dz \,,$$

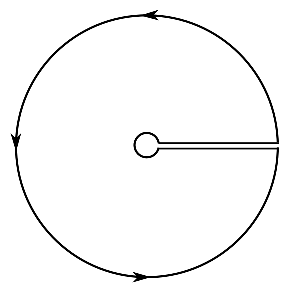
    where ∂R is the boundary of the rectangle $R = \{x+iy \in \mathbb{C} \mid -1 \leq x \leq 2,\ 0 \leq y \leq 1\}$, together with the counterclockwise orientation.

18. State Cauchy's integral theorem for rectangles.

19. What are the main steps in its proof?

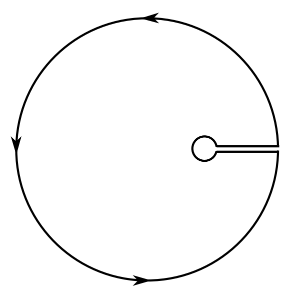
20. Consider the following sketch and assume that the two circles are concentric, their center is at 0, the inner radius is r and the outer radius is R.

    Find a rectangle Q and a C^1-mapping $\varphi : Q \to \mathbb{C}$ which maps the boundary ∂Q of Q to the path depicted in the sketch.



21. (*) Consider the following sketch, and assume that the center of the inner circle is at $x_0 \in \mathbb{R}$, the inner radius is $\hat{r}$ and the center of the outer circle is at 0, the outer radius is $\hat{R} = 1$, and $|x_0| + \hat{r} < 1$.

    Again, find a rectangle $\hat{Q}$ and a C^1-mapping $\hat{\varphi} : \hat{Q} \to \mathbb{C}$ which maps the boundary $\partial\hat{Q}$ of $\hat{Q}$ to the path depicted in the sketch.

    Hint: Use a Möbius transformation of the form

    $$z \mapsto \frac{uz + \sqrt{u^2 - 1}}{\sqrt{u^2 - 1}\, z + u} \text{ for some } u \in \mathbb{R} \text{ with } |u| > 1$$

    to map the inner circle to a circle centered at 0, while mapping the outer circle to itself.

    Another option is to linearly interpolate in the radial direction between a translation by x_0 and the identity.

22. Express $\operatorname{cr}(z_1, z_4, z_2, z_3)$ in terms of $q := \operatorname{cr}(z_1, z_2, z_3, z_4)$.

23. Calculate
    $$\int_{|z-1|=1} \frac{1}{1+z^2}\, dz\,.$$

24. Calculate
    $$\int_{|z-i|=1} \frac{1}{1+z^2}\, dz$$
    using a partial fraction decomposition.

25. Calculate
    $$\int_{|z-i|=1} \frac{1}{1+z^2} \frac{1}{2-z^2}\, dz$$
    by applying Cauchy's integral formula and the partial fraction decomposition from above.

26. Determine
$$\int_{|z|=3} \frac{e^{i\pi z^2}}{1+z^2}\frac{z^4}{2-z^2}\,dz\,.$$

27. Calculate
$$\int_{|z-(1+i)|=2} \frac{e^{i\pi z^2}}{1+z^2}\frac{z^4}{2-z^2}\,dz\,.$$

28. For k in $\mathbb{Z}$, determine
$$\int_{|z|=99} \frac{e^z}{z^k}\,dz\,.$$

29. Show that
$$\int_{|z|=\pi} \frac{\sin(z)}{z^5(z^2+16)}\,dz = \int_{|z-i|=2} \frac{\sin(z)}{z^5(z^2+16)}\,dz\,.$$

30. Find a domain $U \subseteq \mathbb{C}$ and two closed curves α, β in U with common start-/endpoint p such that α and β are freely homotopic, but *not* homotopic through a homotopy which fixes the endpoints.

31. Let f, g be entire functions such that $|f(z)| \leq |g(z)|$ for all z in $\mathbb{C}$. Show that there exists a in $\mathbb{C}$ such that $f = a\,g$.

32. Find two holomorphic functions f and g such that
$$\{z \in \mathbb{C} \mid f(z) = g(z)\}$$
has an accumulation point, but $f \neq g$.

33. Let $f : U \to \mathbb{C}$ be holomorphic, and let z_0 be an isolated singularity of f. Then z_0 is
- a removable singularity
  $\Leftrightarrow$ ____
- a pole of order n
  $\Leftrightarrow$ ____
- an essential singularity
  $\Leftrightarrow$ ____

34. Determine the singularities of the following functions and classify them:

(i) $f(z) = \dfrac{\sin(z)}{z^n}$ (ii) $g(z) = \dfrac{1-\cos z}{\sin z}$ (iii) $h(z) = \cos(z + \dfrac{1}{z})$

35. Let
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$
be the Laurent series development of f around z_0.

Then its principal part is ________________________ and its holomorphic part is

________________________ . It converges for ____________________________ ,

and the coefficients satisfy $c_n =$ ________________________ .

36. Give a Laurent series for

$$f(z) = \frac{2}{z^2 - 4z + 3}$$

on each of the following three annuli:

(i) $A_1 = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ (ii) $A_2 = \{z \in \mathbb{C} \mid 1 < |z| < 3\}$ (iii) $A_3 = \{z \in \mathbb{C} \mid 3 < |z|\}$.

37. What is the statement of the Casorati–Weierstraß theorem?

38. When are two function elements (f, U) and (g, V) called equivalent at a point z_0?

39. What is an analytic continuation (as a sequence/along a curve)?

40. For solutions to equations of which type do you know that their analytic continuations solve them as well?

41. How is the contour integral $\int_\gamma f(z)dz$ defined along a continuous curve γ?

42. Let $U \subseteq \mathbb{C}$ be open, z_0 in U. What is $\pi_1(U, z_0)$?

43. Let $U \subseteq \mathbb{C}$ be open. A 1-chain in U is a ____ .

If ____, then c is said to be closed.

It is called zero-homologous if ____ .

44. Let $U \subseteq \mathbb{C}$ be open, let $z_0 \in U$, let $c : [0,1] \to U$ be a curve, and let $f : U \to C$ be holomorphic. Decide whether the following statements are true or false.

    1. If U is connected, then U is path-connected.
    2. The curve c is null-homotopic if and only if it is zero-homologous.
    3. If c is zero-homologous, then it is closed.
    4. If c is zero-homologous, then it is null-homotopic.
    5. If c is closed and null-homotopic, then it is zero-homologous.
    6. If U is simply connected, then c is zero-homologous.
    7. If c is zero-homologous, then $\int_c f(z)\,dz = 0$.
    8. If U is simply connected, then $f(U)$ is also simply connected.
    9. If c is zero-homologous, then $f \circ c$ is null-homotopic in $f(U)$.
    10. If c is null-homotopic, then $f \circ c$ is zero-homologous in $f(U)$.
    11. The fundamental group $\pi_1(U, z_0)$ is Abelian.
    12. The first homology group $H_1(U, \mathbb{Z})$ is Abelian.

45. What is the statement of the residue theorem?

46. Let $U \subseteq \mathbb{C}$ be a domain, and let $z_0 \in U$. Let $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Show:

   1. If f has a pole of order one in z_0, then

   $$\operatorname{res}(f, z_0) = \lim_{z \to z_0} (z - z_0)\, f(z).$$

   2. If $g, h : U \to \mathbb{C}$ are holomorphic, $g(z_0) \neq 0$, and h has a simple zero at z_0, then

   $$\operatorname{res}\left(\frac{g}{h}, z_0\right) = \frac{g(z_0)}{h'(z_0)}.$$

   3. If f has a pole of order k at z_0, then

   $$\operatorname{res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}((z - z_0)^k f(z)).$$

47. Calculate $\operatorname{res}(z \mapsto \frac{\cot z}{z^2(z+1)}, \pi)$.

48. Calculate the integral $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx$ using the residue theorem.

49. (Exam preparation.) Select a theorem or topic that was covered in the course and ask a partner to explain it to you. Discuss the explanation. Then switch roles and repeat!