Fluid Mechanics

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1 Submanifolds and Lie Groups

1.1 Submanifolds

We start with some basic definitions.

Definition 1. *Let $W, U \subset \mathbb{R}^n$ be open subsets. Then $f : W \to U$ is called a* diffeomorphism *if*

- *f is C^∞ (smooth)*
- *f is bijective*
- *f^{-1} is C^∞*

Definition 2. *A subset $M \subset \mathbb{R}^k$ is called an n-dimensional smooth* submanifold *if for every point $p \in M$ there is an open neighbourhood $W \subset \mathbb{R}^k$, $p \in W$ and a diffeomorphism*

$$\varphi : W \to U \subset \mathbb{R}^k = \mathbb{R}^{k-n} \times \mathbb{R}^n \tag{1}$$

such that

$$\varphi(W \cap M) = \{0\} \times \mathbb{R}^n \cap U \tag{2}$$

Theorem 1. *For $M \subset \mathbb{R}^k$ the following are equivalent*

a) M is an n-dimensional submanifold

b) For each $p \in M$ there is an open neighbourhood $W \subset \mathbb{R}^k$, $p \in W$ and $g : W \to \mathbb{R}^{n-k}$, such that $W \cap M = g^{-1}(\{0\})$ and $g'(0) \in \mathbb{R}^{k-n \times k}$ has rank $k - n$ for all $q \in W$.

c) After applying a permutation of coordinates there are open sets $U \subset \mathbb{R}^n$, $W \subset \mathbb{R}^k$ with $p \in W$ and a smooth map $f : U \to \mathbb{R}^{k-n}$ such that

$$W \cap M = \{(x, f(x)) \mid x \in U\}$$

d) For each $p \in M$ there is an open neighbourhood $W \in \mathbb{R}^k$, an open set $U \subset \mathbb{R}^n$ and a diffeomorphism $\psi : U \to W \cap M$ such that for all $x \in U$ the matrix $\psi'(x) \in \mathbb{R}^{k \times n}$ has rank n. This means: "locally a submanifold can be parameterized".

Proof. (Sketch)
b) $\Rightarrow$ c): implicit function theorem
c) $\Rightarrow$ a): simply project it onto $\mathbb{R}^n$ i.e

$$\varphi(x, f(x)) = x \tag{3}$$

a) $\Rightarrow$ d) define $U := \varphi(W) \cap \{0\} \times \mathbb{R}^n$ and set

$$\psi(x) := \varphi^{-1}(0, x) \tag{4}$$

d) ⇒ c) Apply the inverse function theorem to $\Pi \circ \psi$ where

$$\begin{aligned} \Pi &: \mathbb{R}^k \to \mathbb{R}^n \\ \Pi(x_1, \dots x_k) &= (x_{i_1} \dots x_{i_n}) \\ \{i_1, \dots i_n\} &\subset \{1, \dots k\} \end{aligned} \tag{5}$$

c) ⇒ b) Choose simply

$$\begin{aligned} g &: W \to \mathbb{R}^{k-n} \\ q &\mapsto (q_{n+1}, \dots, q_k) - (f_1(q_1, \dots, q_n), \dots, f_n(q_1, \dots, q_n)) \end{aligned} \tag{6}$$

Then $g'(q)$ has maximal rank since f is assumed to be smooth. □

Definition 3. *Let $M \subset \mathbb{R}^k$ be an n-dimensional submanifold and $p \in M$. Then*

$$T_pM := \{\gamma'(0) \mid \gamma : (\varepsilon, \varepsilon) \to M \text{ smooth }, \gamma(0) = p\} \tag{7}$$

is called tangent space *at p.*

Theorem 2. *T_pM is an n-dimensional linear subspace of $\mathbb{R}^k$.*

Definition 4. *A subset $M \subset \mathbb{R}^n$ is called a* compact domain with smooth boundary, *if for every $p \in \partial M$ there is an open neighbourhood $W \subset \mathbb{R}^n$ and a diffeomorphism*

$$\varphi : W \to U \subset \mathbb{R}^n$$

such that

$$\varphi(W \cap \partial M) = U \cap \{(0, x) | x \in \mathbb{R}^{n-1}\} \tag{8}$$

Definition 5. *Let $M \subset \mathbb{R}^n$ be a subset with the additional requirement that the interior $\mathring{M}$ is dense in M. Then $f : M \to \mathbb{R}^k$ is called* smooth *if there is an open set $U \supset M$ and a smooth extention $\hat{f} : U \to \mathbb{R}^k$ with $\hat{f}|_M = f$.*

Note that we can define all partial derivatives

$$\frac{\partial^{|\alpha|}}{\partial\alpha_1 \cdots \partial\alpha_k} f(p) := \frac{\partial^{|\alpha|}}{\partial\alpha_1 \cdots \partial\alpha_k} \hat{f}(p)$$

independent of the choice of $\hat{f}$ since the partial derivatives of $\hat{f}$ are continuous.

Definition 6. *Let $\hat{M}, M \subset \mathbb{R}^n$ be compact domains with smooth boundary, then a map $g : \hat{M} \to M$ is called an* orientation preserving diffeomorphism *if*

1. *g is smooth in the sense of Definition 5*
2. *g is bijective*
3. *$g^{-1} : M \to \hat{M}$ is C^∞*
4. *$\det g'(p) > 0$ for all $p \in \hat{M}$*

1.2 Lie Groups

Firstly we recall that $GL(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}$ is a group with respect to matrix multiplication and neutral element I.

Definition 7. *A subset $G \subset GL(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$ ia called a* Lie group *if it is a submanifold of $\mathbb{R}^{n\times n}$ and a subgroup of $GL(n,\mathbb{R})$. i.e.*

$$\begin{aligned} A, B \in G &\Rightarrow AB \in G \\ A \in G &\Rightarrow A^{-1} \in G \end{aligned} \tag{9}$$

Definition 8. *Let $G \subset \mathbb{R}^{n\times n}$ be a Lie group. Then*

$$\mathfrak{g} := T_I G \tag{10}$$

is called the Lie algebra *of G.*

Theorem 3. *Let $G \subset \mathbb{R}^{n\times n}$ be a Lie group and $\mathfrak{g} := T_I G$ its Lie algebra. Then for every $A \in G$ the tangent space $T_A G$ is given by*

$$T_A G = \{AX \mid X \in \mathfrak{g}\} = \{XA \mid X \in \mathfrak{g}\} \tag{11}$$

Proof. Take $X \in \mathfrak{g}$ and define the map

$$B : (\varepsilon, \varepsilon) \to G \quad \text{with} \quad B(0) = I \ , \ B'(0) = X \tag{12}$$

Then the map $C : (\varepsilon, \varepsilon) \to G$ defined by $C := AB$ satisfies

$$\begin{aligned} C(0) &= AB(0) = AI = A \\ C'(0) &= AB'(0) = AX \end{aligned}$$

On the other hand take $Y \in T_A G$ and consider the curve

$$C : (\varepsilon, \varepsilon) \to G \quad \text{with} \quad C(0) = A \ , \ C'(0) = Y \tag{13}$$

Then define $B : (\varepsilon, \varepsilon) \to G$ by $B = A^{-1}C$. Obviously $B(0) = I$ and $B'(0) \in T_I G = \mathfrak{g}$. $\square$

Definition 9. *Let $G \in \mathbb{R}^{n\times n}$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Then*

$$\begin{aligned} \exp : \mathfrak{g} &\to \mathbb{R}^{n\times n} \\ X &\mapsto A(1) \end{aligned} \tag{14}$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ solves the initial value problem

$$\begin{cases} A(0) = I \\ A'(t) = A(t)X \end{cases} \tag{15}$$

is called the exponential map.

Theorem 4. *Let $G \in \mathbb{R}^{n\times n}$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Then* *a) For all $X \in \mathfrak{g}$ we have* $\exp(X) \in G$.

b) For $X \in \mathfrak{g}$ the map

$$\mathbb{R} \ni t \mapsto \exp(tX) =: A(t) \tag{16}$$

solves $A'(t) = A(t)X = XA(t)$

c)

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \tag{17}$$

d)

$$\exp((t+s)X) = \exp(tX)\exp(sX) \tag{18}$$

Proof.
a) We start with an arbitrary $X \in \mathfrak{g}$ and consider $\exp(tX)$. Due to the Picard-Lindelöf theorem we find ε such that $\exp(tX) = A(t)$ exists uniquely for $t \in (-\varepsilon, \varepsilon)$. Now let $B \in G$ be the group element which satisfies $B = A(\frac{\varepsilon}{2})$. Hence $A'(\frac{\varepsilon}{2}) = BX \in T_B G$. We now define

$$\begin{aligned} A_1 : (-\varepsilon, \varepsilon) &\to G \\ t &\mapsto A_1(t) := A(t)B \end{aligned} \tag{19}$$

then A_1 solves the initial value problem

$$\begin{cases} A_1(0) = B \\ A_1'(t) = A_1(t)X = A(t)BX \end{cases} \tag{20}$$

Indeed $A_1(t)$ exists uniquely for $t \in (-\varepsilon, \varepsilon)$, since $A(t) \in G$. We proceed as above and construct the iteration

$$A, A_1, A_2, A_3, \ldots$$

Therefore $A(t) \in G$ for all $t \in \mathbb{R}$.
c) We define

$$\tilde{\exp}(X) = I + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots$$

$$\tilde{A}(t) := \tilde{\exp}(tX) = I + tX + t^2\frac{X^2}{2} + \cdots$$

Hence $\tilde{A}(0) = I$ and $\tilde{A}'(t) = X + t\frac{X^2}{1!} + t^2\frac{X^3}{2!} + \cdots = X\tilde{A}(t) = \tilde{A}(t)X$. By the uniqueness part of Picard-Lindelöf we have $\tilde{A} = A$ and finally

$$\tilde{\exp}(X) = \tilde{A}(1) = A(1) = \exp(X)$$

b) is proven by c)
d) Fix $s \in \mathbb{R}$ and define $B, \tilde{B} : \mathbb{R} \to G$ by

$$\begin{aligned} B(t) &:= \exp((t+s)X) \\ \tilde{B}(t) &:= \exp(tX)\exp(sX) \end{aligned}$$

Then $B(0) = \tilde{B}(0) = \exp(sX)$ and

$$\begin{aligned} B'(t) &= XB(t) \\ \tilde{B}'(t) &= X\tilde{B}(t) \end{aligned}$$

Due to the uniqueness part of Picard-Lindelöf one has $B = \tilde{B}$. □

Definition 10. *For $X \in \mathfrak{g}$ the curve*

$$t \mapsto \exp(tX) \in G \tag{21}$$

is called a one-parameter subgroup *of G. (actually a group homomorphism:$\mathbb{R} \to G$)*

Example 1. *Consider the orthonogal group $O(n) := \{A \in \mathbb{R}^{n\times n} \mid A^TA = I\}$ and the curve $t \mapsto A(t) \in O(n)$ where $t \in (-\varepsilon, \varepsilon)$, $A(0) = I$ and $X := A'(0)$. By definition of $O(n)$ we have*

$$\underbrace{A'(0)^T}_{=X^T}\underbrace{A(0)}_{=I} + \underbrace{A(0)^T}_{=I}\underbrace{A'(0)}_{=X} = 0 \;\Leftrightarrow\; X^T + X = 0$$

This shows that $so(n) := \mathfrak{g} \subset \{X \in \mathbb{R}^{n\times n} \mid X^T + X = 0\}$
On the other hand, if $X^T + X = 0$ and $A(t) = \exp(tX)$. Then

$$(A^TA)' = (XA)^TA + A^TXA = A^TX^TA + A^TXA = -A^TXA + A^TXA = 0$$

Hence $A^TA = \text{const}$ and by continuity $A^TA = I$ for all $t \in \mathbb{R}$ since $A^TA(0) = I$.

Definition 11. *Let G be a Lie group, M a manifold. Then a smooth map*

$$\begin{aligned} G \times M &\to M \\ (g, x) &\mapsto g.x \end{aligned} \tag{22}$$

is called a group action *if*

a)

$$I.x = x \qquad \forall x \in M \tag{23}$$

b) for all $A, B \in G$, $x \in M$ hold

$$(AB).x = A.(B.x) \tag{24}$$

Example 2. *Let $M := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ be the unit ball and $G = O(n)$. Then $A.x$ is just matrix multiplication.*

2 Setup of Fluid Mechanics

Definition 12. *Let $\hat{M}, M \subset \mathbb{R}^n$ be compact domains with smooth boundary. Then we define the following useful map*

$$g_t : [t_0, t_1] \times \hat{M} \to M \tag{25}$$

where

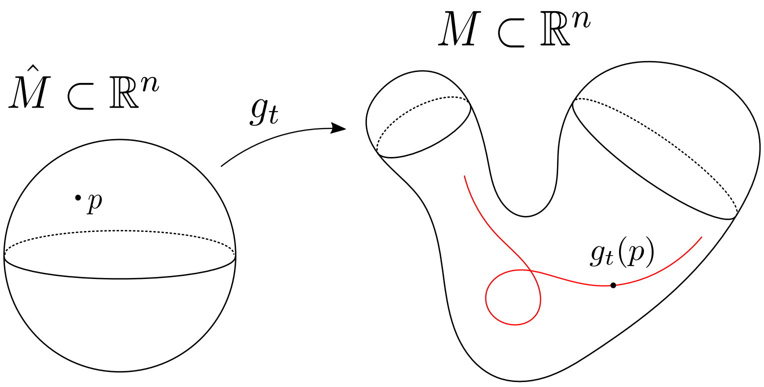
1. *For each $t \in [t_0, t_1]$*

$$g_t : \hat{M} \to M$$

   is an orientation preserving diffeomorphism.

2. *g_t itself is smooth in the sense of Definition 5.*

Intuition 1. *For each fluid molecule $p \in \hat{M}$ its position at time $t \in [t_0, t_1]$ is given by $g_t(p) \in M$.*



Later on we might drop bijectivity (think of waves and air) and very rarely we will drop compactness, because a huge container (ocean) can be seen as a infinitely long container.

Definition 13. *In the space of smooth maps $C^\infty(\hat{M}, \mathbb{R}^n)$ we define the set*

$$\mathcal{M} := \{g : \hat{M} \to M \mid g \text{ orientation preserving diffeomorphism}\} \tag{26}$$

We think about $\mathcal{M}$ as a kind of "submanifold“.

Definition 14. *A map*

$$\begin{aligned}[t_0, t_1] &\to \mathcal{M} \\ t &\mapsto g_t\end{aligned} \tag{27}$$

is called a smooth curve in $\mathcal{M}$ *if the map*

$$\begin{aligned} [t_0, t_1] \times \hat{M} &\to M \\ (t, p) &\mapsto g_t(p) \end{aligned} \tag{28}$$

is smooth.

Definition 15. *For $g \in \mathcal{M}$ we define the tangent space $T_g\mathcal{M}$ by*

$$T_g\mathcal{M} := \{W : \hat{M} \to \mathbb{R}^n \text{ smooth} \mid W(p) \in T_{g(p)}\partial M \ \text{ if } \ p \in \partial\hat{M}\} \tag{29}$$

If $t \mapsto g_t \in \mathcal{M}$, $t \in [t_0, t_1]$ is smooth then we define $\dot{g}_t \in T_{g_t}\mathcal{M}$ by

$$\dot{g}_t(p) := \frac{d}{d\tau}\Big|_{\tau=t} g_\tau(p) \tag{30}$$

Definition 16. *For a smooth family*

$$(-\varepsilon, \varepsilon) \ni t \mapsto h_t \in \mathcal{M} \tag{31}$$

with $h_0 = g$ we define

$$\dot{h}_0 \in T_gM \tag{32}$$

by

$$\dot{h}_0(p) := \frac{\partial h_t(p)}{\partial t}\Big|_{t=0} \tag{33}$$

Theorem 5. *For every $W \in T_g\mathcal{M}$ there is a map*

$$\mathbb{R} \ni t \mapsto h_t \in \mathcal{M} \tag{34}$$

such that

$$W = \dot{h}_0$$

Definition 17.

$$\mathrm{Diff}_0(M) := \{h : M \to M \mid h \ \text{ orientation preserving diffeomorphism}\} \tag{35}$$

This is a group under composition and the neutral element is id_M.

$$\mathrm{diff}_0(M) := \{v : M \to \mathbb{R}^n \ \text{ smooth} \mid q \in \partial M \Rightarrow v_q \in T_q\partial M\} \tag{36}$$

Theorem 6. *For every $g \in \mathcal{M}$ we have the vector space isomorphism*

$$\begin{aligned} \mathrm{diff}_0(M) &\to T_g\mathcal{M} \\ v &\mapsto v \circ g \end{aligned} \tag{37}$$

Proof. Since $v \in \mathrm{diff}_0(M)$ the composition $v \circ g : \hat{M} \to \mathbb{R}^n$ is smooth. By definition $g(p) \in \partial M$ for $p \in \partial \hat{M}$ and therefore $W_p = v_{g(p)} \in T_{g(p)}\partial M$.

In order to construct the inverse linear map we take $W \in T_g M$ and define $v := W \circ g^{-1}$. At the boundary we have for $q \in \partial M \Rightarrow g^{-1}(q) \in \partial \hat{M}$ by definition and therefore

$$v_q = W_{g^{-1}(q)} \in T_q \partial M$$

□

Theorem 7. *For every $v \in \mathrm{diff}_0(M)$ there is a smooth family*

$$\mathbb{R} \ni t \mapsto h_t \in \mathrm{Diff}_0(M) \tag{38}$$

such that

1. $h_0 = id_M$
2. $\dot{h}_t(q) = v_{h_t(q)} \quad \forall q \in M, \quad \forall t \in \mathbb{R}$
3. $h_{t+s} = h_t \circ h_s$

Proof. Firstly we look at the boundary. For each $q \in \partial M$ we choose a coordinate chart (V_q, φ_q) where $V_q \subset \partial M$ is an open neighbourhood of q and $\varphi_q : V_q \to U_q \subset R^{n-1}$ is assumed to be a diffeomorphism, since ∂M is a smooth boundary. Since ∂M is compact, the smooth vector field $v|_{\partial M}$ is bounded and therefore Lipschitz continuous. Thus we can apply the Picard-Lindelöf theorem to the vector field

$$u_{\varphi_q(x)} = d_x \varphi_q(v_x) \qquad x \in \partial M \tag{39}$$

This means there is an open subset $\tilde{U}_q \subset U_q$ and $\varepsilon_q > 0$ such that for any $p \in \tilde{U}_q$ there is an integral curve

$$\gamma : (-\varepsilon_q, \varepsilon_q) \to U_q \tag{40}$$

with

$$\begin{cases} \gamma_p(0) = p \\ \dot{\gamma}_p(t) = u_{\gamma_p(t)} \end{cases} \tag{41}$$

Moreover let $\tilde{V}_q := \varphi_q^{-1}(\tilde{U}_q)$ be the corresponding neighbourhood on ∂M. By compactness there are $q_1, \ldots, q_k \in \partial M$ such that $\tilde{V}_{q_1}, \ldots, \tilde{V}_{q_k}$ cover ∂M. Hence one can define a global $\varepsilon := \min\{\varepsilon_{q_1}, \ldots, \varepsilon_{q_k}\}$.
We now jump back to the boundary of M. For every $q \in \partial M$ there is a solution

$$\eta_q : (-\varepsilon, \varepsilon) \to \partial M \tag{42}$$

with

$$\begin{cases} \eta_q(0) = q \\ \dot{\eta}_q(t) = v_{\eta_q(t)} \end{cases} \tag{43}$$

Since $q_1 := \eta_q(\frac{\varepsilon}{2})$ can be used as starting point for

$$\eta_{q_1} : (-\varepsilon, \varepsilon) \to \partial M$$

η_q is extendable to

$$\eta_q : \mathbb{R} \to \partial M$$

We now define

$$h_t : \partial M \to \partial M \tag{44}$$

by

$$h_t(q) := \eta_q(t) \tag{45}$$

for all $t \in \mathbb{R}$.

- By assumption we have a smooth dependency on the initial data, hence h_t is smooth.
- Clearly $h_{t+s} = h_t \circ h_s$ and in particular $h_{-t} \circ h_t = h_0 = id_M$. Hence the inverse $h_t^{-1} := h_{-t}$ exists and h_t is bijective.

Putting it all together, we eventually obtain that h_t is a diffeomorphism.

Now we do the same on M. Let $\check{M} \supset M$ be an open bounded set containing M. Extend the vector field v to $\check{v}$ smoothly on the closure $\overline{\check{M}}$. Then $\check{v}$ is Lipschitz. Consider the open cover

$$\check{M} = \bigcup_{\check{q} \in \check{M}} \check{V}_{\check{q}} \tag{46}$$

Again we send the covering to $\mathbb{R}^{n-1}$ via $\check{\varphi}_{\check{q}}$

$$\check{U}_{\check{q}} := \check{\varphi}_{\check{q}}(\check{V}_{\check{q}}) \tag{47}$$

Due to Picard-Lindelöf we find $\tilde{\check{U}}_{\check{q}}$ and $\varepsilon_q > 0$ such that for any $p \in \tilde{\check{U}}_q$ there is an integral curve

$$\gamma : (-\varepsilon_q, \varepsilon_q) \to \check{U}_q \tag{48}$$

By the above procedure we find a global ε and finitely many q_i such that

$$\eta_{q_i} : (-\varepsilon, \varepsilon) \to \tilde{M} \tag{49}$$

with

$$\begin{cases} \eta_q(0) = q \in \tilde{\check{V}}_{\check{q}} := \check{\varphi}_{\check{q}}(\tilde{\check{U}}_{\check{q}}) \\ \dot{\eta}_q(t) = v_{\eta_q(t)} \end{cases} \tag{50}$$

is well defined for all $q \in M$.
If we start at $q \in \partial M$ we stay on ∂M.
If we start in the interior at $q \in \mathring{M}$ then by the uniqueness part of Picard-Lindelöf we never hit ∂M. This implies that we never leave M.

As before we construct

$$h_t : M \to M \quad \forall t \in \mathbb{R} \tag{51}$$

Due to the smooth dependency on the initial data all h_t are smooth. On the otherhand we have $h_{t+s} = h_t \circ h_s$ which gives us the bijectivity. Therefore all h_t are diffeomorphisms. Since h_0 is orientation preserving and by the continuity of the determinant all $h_t \in \mathrm{Diff}_0(M)$ for all t. □

We now say that $\mathrm{Diff}_0(M)$ is like a Lie group with identity element id_M and its Lie algebra "$T_{id_M}\mathrm{Diff}_0(M)$ is equal to $\mathrm{diff}_0(M) =: \mathfrak{g}$. Moreover for $h \in \mathcal{M}$ and $g \in \mathrm{Diff}_0(M)$ we define by "$g.h := g \circ h$" a "*smooth group action*" of $\mathrm{Diff}_0(M)$ on $\mathcal{M}$.

Definition 18. *For $v \in \mathrm{diff}_0(M)$ we define the exponential map*

$$\exp(v) \in \mathrm{Diff}_0(m) \tag{52}$$

by

$$\exp(v)(p) = \gamma(1) \tag{53}$$

where

$$\begin{cases} \gamma(0) = p \\ \dot{\gamma}(t) = v_{\gamma(t)} \end{cases} \tag{54}$$

Indeed, we have: $\exp((t+s)v) = \exp(tv) \circ \exp(sv)$ and $\exp(0) = id_M$.

Suppose now that the volume $vol(U) := \int_U 1$ for $U \in \hat{M}$ can be interpreted as the total mass of the fluid molecules in U.

Suppose $t \mapsto g_t \in \mathcal{M}$ describes the motion of the fluid in M. Then each fluid molecule $p \in \hat{M}$ has velocity $\dot{g}_t(p) \in \mathbb{R}^n$. So we can write the kinetic energy as

$$\frac{1}{2} m |\dot{g}_t|^2 \tag{55}$$

Moreover the total kinetic energy (up to a constant) is

$$E(g_t) = \int_{\hat{M}} |\dot{g}_t|^2 \tag{56}$$

With this as background we can define a metric on $T_{g_t}\mathcal{M}$.

Definition 19. *The map*

$$\begin{aligned} \langle\langle \cdot, \cdot \rangle\rangle : T_{g_t}\mathcal{M} \times T_{g_t}\mathcal{M} &\to \mathbb{R} \\ (g, h) &\mapsto \frac{1}{2} \int_{\hat{M}} \langle g, h \rangle \end{aligned} \tag{57}$$

can be viewed as a "Riemannian metric" on $\mathcal{M}$.

Definition 20. *Suppose* $[a,b] \ni t \mapsto g_t$ *is a smooth curve in* $\mathcal{M}$. *Then a* variation *of* g *is a smooth map*

$$\alpha : (-\varepsilon, \varepsilon) \times [a,b] \times \hat{M} \to M \tag{58}$$

such that for all $(\tau, t) \in (-\varepsilon, \varepsilon) \times [a,b]$ *the map*

$$\begin{aligned} g_{\tau,t} : \hat{M} &\to M \\ p &\mapsto \alpha(\tau, t, p) \end{aligned} \tag{59}$$

is in $\mathcal{M}$.
Moreover the variational vector field $Y_t \in T_{g_t}\mathcal{M}$ *is defined as*

$$Y_t(p) := \frac{\partial \alpha(\tau, t, p)}{\partial \tau}\Big|_{\tau=0} \tag{60}$$

Definition 21. *The* action of a free motion $[a,b] \ni t \mapsto g_t \in \mathcal{M}$ *is defined by*

$$S(g) := \frac{1}{2}\int_a^b \langle\langle \dot{g}_t, \dot{g}_t \rangle\rangle dt \tag{61}$$

Theorem 8. *(First variational formula)*
Let α *be a variation of* $[a,b] \ni t \mapsto g_t \in \mathcal{M}$ *with a variational vector field* Y_t. *Then*

$$\frac{d}{d\tau}\Big|_{\tau=0} S(g_{\tau,\cdot}) = \langle\langle Y_t, \dot{g}_t \rangle\rangle \Big|_{t=a}^{t=b} - \int_a^b \langle\langle Y_t, \ddot{g}_t \rangle\rangle dt \tag{62}$$

Before we prove the theorem we introduce the following notation

$$\frac{\partial f}{\partial \tau} = \mathring{f} \tag{63}$$

Proof.

$$\begin{aligned} \frac{d}{d\tau}\Big|_{\tau=0} S(g_{\tau,\cdot}) &= \int_a^b \langle\langle \underbrace{\dot{\mathring{g}}_{o,t}}_{=:\mathring{\dot{g}}_t}, \underbrace{\dot{g}_{o,t}}_{=:\dot{g}_t} \rangle\rangle dt \\ &= \int_a^b \left(\frac{\partial}{\partial t}\langle\langle \mathring{g}_t, \dot{g}_t \rangle\rangle - \langle\langle \mathring{g}_t, \ddot{g}_t \rangle\rangle\right) dt \\ &= \langle\langle Y_t, \dot{g}_t \rangle\rangle \Big|_{t=a}^{t=b} - \int_a^b \langle\langle Y_t, \ddot{g}_t \rangle\rangle dt \end{aligned}$$

where the last identity is given by the fundamental theorem of calculus. □

Theorem 9. *Let* $[a,b] \ni t \mapsto Y_t \in T_{g_t}\mathcal{M}$ *be smooth as a function on* $[a,b] \times \hat{M}$. *Then there is a variation* $\alpha : \mathbb{R} \times [a,b] \times \hat{M} \to M$ *with the variational vector field* Y_0.

Proof.

$$g_{\tau,t} := \exp(\tau \underbrace{Y_t \circ g_t^{-1}}_{\in \mathrm{diff}_0(m)}) \circ g_t \tag{64}$$

□

Definition 22. *The curve*

$$\begin{aligned} [a,b] &\to \mathcal{M} \\ t &\mapsto g_t \end{aligned} \tag{65}$$

is called a geodesic *in $\mathcal{M}$ if for all variations α of $(t \mapsto g_t)$ with fixed endpoints i.e $g_{\tau,a} =: g_a$ and $g_{\tau,b} = g_b$ for all $\tau \in (-\varepsilon, \varepsilon)$*

$$\frac{d}{d\tau}\Big|_{\tau=0} S(t \mapsto g_{\tau,t}) = 0 \tag{66}$$

"principle of least action"

Theorem 10. *The curve*

$$\begin{aligned} [a,b] &\to \mathcal{M} \\ t &\mapsto g_t \end{aligned}$$

is a geodesic if and only if $\ddot{g}_t = 0$ for all $t \in [a,b]$.

Proof. "$\Leftarrow$" Let α be a variation with fixed endpoints. Then the first variational formula gives us $\frac{d}{d\tau}\Big|_{\tau=0} S(t \mapsto g_{\tau,t}) = 0$ since the endpoints are fixed $\mathring{g}_a = \mathring{g}_b = 0$.

"$\Rightarrow$" Suppose $t \mapsto g_t$ and $\ddot{g}_t \neq 0$. Then there is $t \in (a,b)$ and $p \in \mathring{\hat{M}}$ with $\ddot{g}_t(p) \neq 0$. Choose $\delta > 0$ such that $[t-\delta, t+\delta] \subset (a,b)$ and $\{q \in \mathbb{R}^n \mid |q-p| < \delta\} \subset \mathring{\hat{M}}$. Then for $s \in [a,b]$ define $\hat{Y}_s$ by $\hat{Y}_s := \varphi_s \ddot{g}_s$ where $\varphi_s(q) := \psi(s)\rho(q)$ with

$$\psi : [a,b] \to \mathbb{R}$$

$$\begin{cases} \psi(t) > 0 \\ \psi(s) \geq 0 & \text{for } \; s \in [t-\delta, t+\delta] \\ \psi(s) = 0 & \text{for } \; s \notin [t-\delta, t+\delta] \end{cases}$$

and

$$\rho : M \to \mathbb{R}$$

$$\begin{cases} \rho(p) > 0 \\ \rho(x) \geq 0 & \text{for } \; x \in \{q \in \mathbb{R}^n \mid |q-p| < \delta\} \\ \rho(s) = 0 & \text{for } \; x \notin \{q \in \mathbb{R}^n \mid |q-p| < \delta\} \end{cases}$$

Now define

$$\begin{aligned} \alpha : \mathbb{R} \times [a,b] \times \hat{M} &\to M \\ \alpha(\tau, s, p) &= \exp(\tau Y_s) \end{aligned} \tag{67}$$

where $Y_s = \hat{Y}_s \circ g_s^{-1}$.
Note that Y_s vanishes on ∂M and therefore $Y_s \in \mathrm{diff}_0(M)$.
Due to the first variational formula one obtains

$$0 \overset{geodesic}{=} \frac{d}{d\tau}\Big|_{\tau=0} S(\tau(g_{\tau,\cdot})) = -\int_a^b \int_M \underbrace{\langle \hat{Y}_s, \ddot{g}_s \rangle}_{\underbrace{\varphi_s}_{\geq 0} \; |\ddot{g}_s|^2} < 0$$

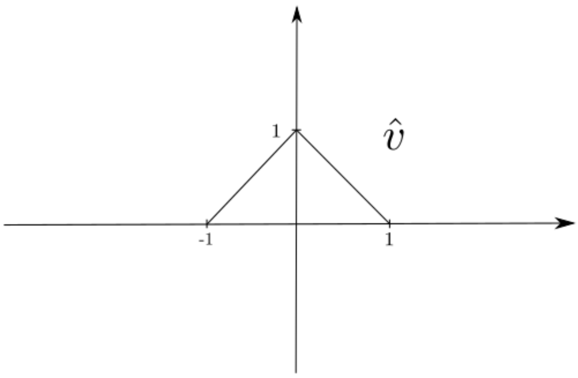
This contradicts $\ddot{g}_t \neq 0$. $\square$

Thus, for a geodesic there is a fixed $\hat{v} : \hat{M} \to \mathbb{R}^n$ such that $\dot{g}_t = \hat{v}$ for all $t \in [a, b]$. So for all $p \in \hat{M}$, $t \in [a, b]$ a geodesic containing the point $g_a(p)$ is the straight line

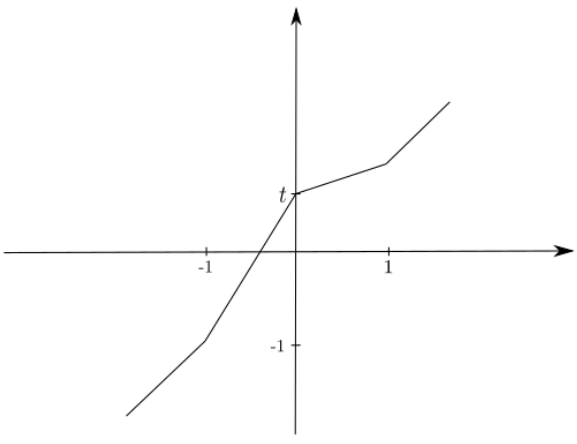
$$g_t(p) = g_a(p) + (t - a)\hat{v}(p) \tag{68}$$

Note that the particles move independently of each other.

Example 3. 1D *shock waves.*
Let $M = \hat{M} = [-2, 2]$ *and* $g_0 = id_M$.



$$g_t(x) = x + t\hat{v}(x) = \begin{cases} x & , \ |x| \geq 1 \\ x + t(1 - |x|) & , \ |x| < 1 \end{cases}$$



Note that g_t *is not an orientation preserving diffeomorphism for* $t > 1$.

3 Natural Potential Energy for Fluids (Gases)

Before we start investigating the model of interacting particles, we consider a single mass point under a potetial and derive Newtons law from the principle of least action.

3.1 Mass Point in $\mathbb{R}^n$

Given a mass point in $\mathbb{R}^n$ with unit mass and a smooth curve

$$\gamma : [a,b] \to \mathbb{R}^n$$

Moreover let $V : \mathbb{R}^n \to \mathbb{R}$ be a potential. Then

- the Kinetic energy at time t is: $\frac{1}{2}|\dot{\gamma}(t)|^2$
- the action for free motion is: $S(\gamma) = \int_a^b \frac{1}{2}|\dot{\gamma}(t)|^2 dt$
- the action under a conservative force is : $S(\gamma) = \int_a^b \frac{1}{2}|\dot{\gamma}(t)|^2 - V(\gamma(t))dt$

Consider the variation of a curve γ with fixed end points $(\gamma_\tau(a) = \gamma(a), \gamma - \tau(b) = \gamma(b))$. Since for a variational vector field Y we have $\mathring{\gamma} = Y$ and hence $Y_a = Y_b = 0$. Therefore

$$\begin{aligned}
\frac{d}{d\tau}\Big|_{\tau=0} S(\gamma_\tau) &= \int_a^b (\langle \dot{\mathring{\gamma}}, \dot{\gamma} \rangle - \langle \operatorname{grad} V \circ \gamma, \mathring{\gamma} \rangle) \\
&= \int_a^b (\langle \mathring{\gamma}, \dot{\gamma} \dot{\rangle} - \langle \mathring{\gamma}, \ddot{\gamma} \rangle - \langle \operatorname{grad} V \circ \gamma, \mathring{\gamma} \rangle) \\
&= \underbrace{\langle Y, \dot{\gamma} \rangle \Big|_a^b}_{=0} - \int_a^b \langle Y, \ddot{\gamma} + \operatorname{grad} V \circ \gamma \rangle
\end{aligned}$$

Since Y is arbitrary the curve γ is a critical point of S with respect to variations with fixed ends if and only if

$$\ddot{\gamma} = - \operatorname{grad} V \circ \gamma \tag{69}$$

3.2 Barotropic Fluid

Firstly, let us recall the change of volume. Assume that $U \subset \hat{M}$, $V \subset M$ and $g : \hat{M} \to M$ is a diffeomorphism. Then

- $vol(g(U)) = \int_U \det g'$
- $vol(U) = \int_{g(U)} \det(g^{-1})'$
- $vol(g^{-1}(V)) = \int_V \det(g^{-1})' = \int_V \frac{1}{\det g'} \circ g^{-1}$

Note that for small U one can approximate

$$vol(V) = vol(g(U)) = \int_U \det g' \approx \det g'(p)\, vol(U)$$

Thus the density of a fluid at q an be defined by

$$\hat{\rho}(q) = \frac{vol(U)}{vol(V)} \approx \frac{1}{\det g'} \tag{70}$$

This justifies the following definition.

Definition 23. *Let $g : \hat{M} \to M$ be an orientation preserving diffeomorphism. Then*

- $$\begin{aligned} \rho : M &\to \mathbb{R} \\ \rho &= \frac{1}{\det g'} \circ g^{-1} \end{aligned} \tag{71}$$

  is called Euclidian density.

- $$\begin{aligned} \hat{\rho} : \hat{M} &\to \mathbb{R} \\ \hat{\rho} &= \frac{1}{\det g'} \end{aligned} \tag{72}$$

  is called Lagrangian density.

More generally we define

1. **Lagrangian viewpoint**: Following each fluid particle $p \in \hat{M}$.
2. **Eulerian viewpoint**: Hover over a fixed point $q \in M$.

Given a function $W : (0, \infty) \to \mathbb{R}$, then one can define for $g \in \mathcal{M}$ the potential V by

$$V(g) := \int_{\hat{M}} W(\hat{\rho}) = \int_M \underbrace{W(\hat{\rho}) \circ g^{-1}}_{W(\underbrace{\hat{\rho} \circ g^{-1}}_{\rho})} \underbrace{\det(g^{-1})'}_{\underbrace{\frac{1}{\det g'} \circ g^{-1}}_{\rho}} = \int_M \rho W(\rho) \tag{73}$$

If we like to apply the principle of least action an expression for the variation of the density is needed.
Unfortunately $\rho : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}$ lives on $(-\varepsilon, \varepsilon) \times M$. Therefore we have to extend our g to

$$\begin{aligned} g : (-\varepsilon, \varepsilon) \times \hat{M} &\to (-\varepsilon, \varepsilon) \times M \\ (\tau, p) &\mapsto (\tau, g_\tau(p)) \end{aligned} \tag{74}$$

and obtain

$$\begin{aligned}\hat{\rho} = \rho \circ g : (-\varepsilon, \varepsilon) \times \hat{M} &\to \mathbb{R} \\ (\tau, p) &\mapsto \rho(g(\tau, p)) = \rho(\tau, g_\tau(p))\end{aligned} \tag{75}$$

For convenience we define the curve

$$\begin{aligned}\gamma_p : (-\varepsilon, \varepsilon) &\to (-\varepsilon, \varepsilon) \times \hat{M} \\ \tau &\mapsto (\tau, p)\end{aligned} \tag{76}$$

Then obviously $g(\gamma_p(\tau)) = (\tau, g_\tau(p))$.

To prepare for the next theorem we need the following

Proposition 11. *Let $A : (-\varepsilon, \varepsilon) \to \mathrm{GL}(n, \mathbb{R})$ be a curve in the invertible $n \times n$ matrices, where $B := A(0)$ and $\mathring{B} = A'(0)$. Then*

$$\frac{d}{d\tau}\Big|_{\tau=0} \det A(\tau) = \det B \ \mathrm{tr}(B^{-1}\mathring{B}) \tag{77}$$

Proof. We introduce the following notation $B^{-1}\mathring{B}e_j = \sum_{i=1}^n c_{ij}e_i$.

$$\begin{aligned}\mathring{\det(Be_1, \dots Be_n)} &= \det(\mathring{B}e_1, Be_2, \dots Be_n) + \dots + \det(Be_1, \dots Be_{n-1}, \mathring{B}e_n) \\ &= \det(BB^{-1}\mathring{B}e_1, Be_2, \dots Be_n) + \dots + \det(Be_1, \dots Be_{n-1}, BB^{-1}\mathring{B}e_n) \\ &= \det B(\underbrace{\det(B^{-1}\mathring{B}e_1, Be_2, \dots Be_n)}_{c_{11}} + \dots + \underbrace{\det(Be_1, \dots Be_{n-1}, B^{-1}\mathring{B}e_n)}_{c_{nn}}) \\ &= \det B \ \mathrm{tr}(B^{-1}\mathring{B})\end{aligned}$$

□

Now we are in a position to state and prove the “continuity equation”.

Theorem 12. Continuity Equation - (Euler)
Let $Y : M \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field on M and let $\rho : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}$ be a density. Then holds

$$\mathring{\rho} + \langle \mathrm{grad}\, \rho, Y \rangle = -\rho \ \mathrm{div} Y \tag{78}$$

or equivalently

$$\mathring{\rho} + \mathrm{div}(\rho\, Y) = 0 \tag{79}$$

Proof. Firstly we compute

$$
\begin{aligned}
\mathring{\hat{\rho}} &= \frac{d}{ds}\Big|_{s=\tau} \rho \circ g \circ \gamma_p(s) \\
&= (\mathring{\rho}, \operatorname{grad} \rho) \circ g \circ \gamma \left(\begin{array}{c|c} 1 & 0 \; \dots \; 0 \\ \hline \mathring{g} \circ \gamma & g' \circ \gamma \end{array} \right) \begin{pmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix} \\
&= (\mathring{\rho}, \operatorname{grad} \rho) \circ g \begin{pmatrix} 1 \\ \mathring{g} \end{pmatrix} \\
&= \mathring{\rho} \circ g + \langle \operatorname{grad} g \circ g, \underbrace{\mathring{g}}_{Y \circ g} \rangle \\
&= (\mathring{\rho} + \langle \operatorname{grad} g, Y \rangle) \circ g
\end{aligned}
$$

On the other hand we have

$$
\mathring{(\det g')} = \mathring{(\det \hat{\rho}^{-1})} = \left(\frac{1}{\rho \circ g} \right)^{\circ} = -\frac{\mathring{(\rho \circ g)}}{(\rho \circ g)^2} = -\frac{\mathring{\hat{\rho}}}{(\rho \circ g)^2}
$$

Using Proposition 11 leads to

$$
\begin{aligned}
-\frac{\mathring{\rho} + \langle \operatorname{grad} g, Y \rangle}{\rho^2} \circ g &= \underbrace{\det g'}_{\frac{1}{\rho \circ g}} \operatorname{tr}((g')^{-1} \underbrace{\mathring{g}'}_{(Y' \circ g) \cdot g'}) \\
&= \frac{1}{\rho \circ g} \underbrace{\operatorname{tr} Y'}_{\operatorname{div} Y} \circ g
\end{aligned}
$$

$$
\Leftrightarrow \quad \mathring{\rho} + \langle \operatorname{grad} \rho, Y \rangle = -\rho \operatorname{div} Y
$$

□

Theorem 13. Continuity Equation - (Lagrange)
Let $Y : M \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field on M and let $\hat{\rho} : (-\varepsilon, \varepsilon) \times \hat{M} \to \mathbb{R}$ be a density on $\hat{M}$. Then holds

$$
\mathring{\hat{\rho}} = -\hat{\rho} \, (\operatorname{div} Y) \circ g \tag{80}
$$

Proof.

$$
\begin{aligned}
\mathring{\hat{\rho}} &= \left(\frac{1}{\det g'} \right)^{\circ} = -\frac{1}{(\det g')^2} (\det g')^{\circ} = \hat{\rho}^2 \det g' \operatorname{tr}(g'^{-1} \mathring{g}') \\
&= \hat{\rho} \operatorname{tr}(g'^{-1} (Y \circ g)' g') = \hat{\rho} \operatorname{tr}(Y') \circ g = \hat{\rho} \, (\operatorname{div} Y) \circ g
\end{aligned}
$$

□

We now look at a variation of the action of a free motion with a variational vector field with fixed end points i.e. $g_a = g_b = 0$. Then

$$\mathring{S} = -\int_a^b \int_{\hat{M}} \langle \mathring{g}, \ddot{g} \rangle \tag{81}$$

Consider a time depending vector field $X : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^n$. Then

$$(X \circ g)^{\cdot} = (\dot{X} + \nabla_v X) \circ g$$

Applying this to $X = v$ yields

$$\ddot{g} = (v \circ g)^{\cdot} = (\dot{v} + \nabla_v v) \circ g \tag{82}$$

Hence

$$\begin{aligned} \mathring{S} &= \int_a^b \int_{\hat{M}} -\langle Y, \dot{v} + \nabla_v v \rangle \circ g \\ &\overset{\rho = \frac{1}{\det g'} \circ g}{=} \int_a^b \int_M -\rho \langle Y, \dot{v} + \nabla_v v \rangle \end{aligned} \tag{83}$$

- **Lagrangian viewpoint**: $\Rightarrow \ddot{g} = 0$.
- **Eulerian viewpoint**: watch $v_t(q) \Rightarrow \dot{v} + \nabla_v v = 0$.

The next step is to compute the infinitesimal variation of the potential energy.

Theorem 14. *Let*

$$\begin{aligned} V : \mathcal{M} &\to \mathbb{R} \\ g &\mapsto \int_{\hat{M}} W(\hat{\rho}) \end{aligned}$$

be the potential energy. Where $W : (0, \infty) \to \mathbb{R}$ is a given function and $\hat{\rho} : \hat{M} \to \mathbb{R}_+$ is the density on $\hat{M}$. Then the variation of V is

$$\mathring{V} = \int_M \rho \, \langle Y, \mathrm{grad}(W \circ \rho + \rho W' \circ \rho) \rangle \tag{84}$$

Proof. Firstly recall the following two identities

- Divergence theorem

$$\int_M \mathrm{div} X = \int_{\partial M} \langle X, N \rangle = 0 \quad \forall X \in \mathrm{diff}_0(M) \tag{85}$$

- Let $f : M \to \mathbb{R}$ be a real valued function, then

$$\mathrm{div}(fY) = \langle \mathrm{grad}\, f, Y \rangle + f \mathrm{div} Y \tag{86}$$

As we have seen before $V(g) = \int_M \rho\, W(\rho)$. Hence

$$\mathring{V} = \int_M \mathring{\rho}\, W(\rho) + \rho\, W'(\rho)\, \mathring{\rho} \tag{87}$$

Plugging in the continuity equation leads to

$$\begin{aligned}\mathring{V} &= \int_M -(\rho\, \mathrm{div} Y + \langle \mathrm{grad}\, \rho, Y\rangle)(W(\rho) + \rho\, W'(\rho)) \\ &\overset{(86)}{=} \int_M -\mathrm{div}(\rho Y)\ (W(\rho) + \rho\, W'(\rho))\end{aligned}$$

Using again (86) by setting $f := -(W(\rho) + \rho\, W'(\rho))$ and (85) completes the proof. □

We now consider the variation of $S(g) = \int_a^b \left[\frac{1}{2}\int_M \rho_t |v_t|^2 - V(g_t)\right] dt$

$$\mathring{S} = \int_a^b \int_M \rho\langle Y, \dot{v} + \nabla_v v - \mathrm{grad}(W(\rho) + \rho W'(\rho))\rangle \tag{88}$$

Then $\mathring{S} = 0$ for all variations Y with fixed end points if and only if

$$\dot{v} + \nabla_v v = -\,\mathrm{grad}\, p \tag{89}$$

where $p := -(W(\rho) + \rho W'(\rho))$ is called *pressure.*

Theorem 15. *The total energy $H := E + V$ is constant.*

Proof. Consider the kinetic energy $E = \frac{1}{2}\int_M \rho\langle v, v\rangle$ on M. From the Eulerian continuity equation we know

$$\mathring{\rho} = -\mathrm{div}(\rho\, Y)$$

Then

$$\begin{aligned}\mathring{E} &= \int_M \left(-\frac{1}{2}\mathrm{div}(\rho v)\langle v, v\rangle - \rho\langle v, \dot{v}\rangle\right) \\ &\overset{(89)}{=} \int_M \left(-\frac{1}{2}\mathrm{div}(\rho v)\langle v, v\rangle - \rho\langle v, \nabla_v v + \mathrm{grad}\, p\rangle\right) \\ &\overset{(86)}{=} \int_M \left(-\frac{1}{2}\mathrm{div}(\rho v\langle v, v\rangle) + \frac{1}{2}\langle \rho v, \mathrm{grad}\langle v, v\rangle\rangle - \langle \rho v, \nabla_v v + \mathrm{grad}\, p\rangle\right) \\ &= \int_M (\langle \rho v, \nabla_v v\rangle - \langle \rho v, \nabla_v v + \mathrm{grad}\, p\rangle) \\ &\overset{(89)}{=} -\int_M \rho\langle v, \mathrm{grad}\, p\rangle\end{aligned}$$

Comparing this with the variation of the potential energy (84) proves the statement. □

Remark 1. *Let $\lambda, \mu \in \mathbb{R}$ be real constants. Moreover we define $\tilde{W}$ by*

$$\tilde{W}(\rho) := W(\rho) + \lambda + \frac{\mu}{\rho}$$

Then the corresponding dynamics coincide.
We simply compute

$$\tilde{p} = \tilde{W}(\rho) + \rho\tilde{W}'(\rho) = p + \lambda + \frac{\mu}{\rho} - \rho\frac{\mu}{\rho^2} = \rho + \lambda$$

$$\Rightarrow \quad \operatorname{grad}\tilde{p} = \operatorname{grad} p$$

Example 4. *Ideal gas $W(\rho) = \rho$. Then $p = \rho + \rho \cdot 1 = 2\rho$*

4 Incompressible Fluids

Definition 24. *A map $g : \hat{M} \to M$ is called* volume preserving diffeomorphism *if*

1. *g is a diffeomorphism*
2. *$vol(\hat{U}) = vol(g(\hat{U}))$ for all subsets $\hat{U} \subset \hat{M}$*

Definition 25. *The map $g : \hat{M} \to M$ is called* incompressible fluid *if g is a volume preserving diffeomorphism.*

Intuitively it seems clear that the corresponding vector field is divergence free, because the volume of any subset is preserved. But we are going to prove this rigorously.

As above we define

$$\tilde{\mathcal{M}} := \{g : \hat{M} \to M \mid \det g' = 1\} \subset \mathcal{M} \tag{90}$$

and the Lie group

$$\operatorname{SDiff}(M) := \{h : M \to M \mid \det h' = 1\} \tag{91}$$

Proposition 16. *The corresponding Lie algebra is*

$$\operatorname{sdiff}(M) := \{X \in \Gamma(TM) \mid \langle X, N\rangle = 0 \text{ on } \partial M \text{ , } \operatorname{div}X = 0\} \tag{92}$$

Proof. Set $X_t(q) = \frac{d}{dt}\Big|_{t=0} h_t(q)$. Since $\rho_t \equiv 1$ the Eulerian continuity equation (79) becomes

$$\underbrace{\dot{\rho}}_{=0} + \operatorname{div}(\rho X) = 0 \quad \Leftrightarrow \quad \operatorname{div}X = 0$$

Conversely take $X \in \operatorname{sdiff}(M)$ and define for all $t \in \mathbb{R}$

$$h_t := \exp(tX) \in \operatorname{Diff}_0(M)$$

Due to the Lagrangian continuity equation (80)

$$\mathring{\hat{\rho}}_t = -\hat{\rho}_t \underbrace{(\text{div} X_t)}_{=0} \circ h_t$$

one obtains that $\hat{\rho}_t$ is constant. Since $h_0 = id$ and $\hat{\rho}_t = \frac{1}{\det h_t'}$ one can conclude that $\hat{\rho}_t \equiv 1$. □

Again we define the incompressible motion by the priciple of least action

$$S(g_t) = \int_a^b \int_{\hat{M}} \frac{1}{2}|\dot{g}_t|^2 \qquad g_t \in \tilde{\mathcal{M}}$$

We already know that

$$\mathring{S} = -\int_a^b \int_{\hat{M}} \langle \ddot{g}_t, Y_t \circ g_t \rangle dt$$

The only difference now is that $\mathring{S} = 0$ for all Y_t such that

$$\begin{cases} Y_a = Y_b = 0 \\ Y_t \in \text{sdiff}(M) \end{cases} \tag{93}$$

The same reasoning as before now yields

$$0 = \int_{\hat{M}} \langle \ddot{g}_t, Y_t \circ g_t \rangle = \int_M \langle \dot{v} + \nabla_v v, Y_t \rangle \qquad \forall Y_t \ \in \text{sdiff}(M)$$

i.e. $\dot{v} + \nabla_v v \in \text{sdiff}(M)^\perp$
In summary can be said

Theorem 17. *The action S is critical if and only if*

$$\dot{v} + \nabla_v v \in \text{sdiff}(M)^\perp \quad \text{for all} \quad t \in [a, b]$$

Before we start to characterize the orthonogal complement of $\text{sdiff}(M)$, we introduce the notation of *musical isomorphisms.*

Definition 26. *Let M be a smooth Remanian manifold with metric $\langle \cdot, \cdot \rangle$. Then the pointwise defined operator*

$$\begin{aligned} {}^\flat : T_pM &\to \Omega_p^1(M) = T_p^*M \\ X_p &\mapsto \langle X_p, \cdot \rangle_p \end{aligned} \tag{94}$$

is called flat operator *and its inverse*

$$^\sharp : \Omega_p^1(M) \to T_pM \tag{95}$$

such that

$$\langle \omega_p^\sharp, \cdot \rangle_p = \omega_p \tag{96}$$

is called sharp operator.

Moreover we define $W_t \in \Gamma(TM)$ by

$$\ddot{g}_t = W_t \circ g_t \tag{97}$$

Then the action S is critical if and only if

$$0 = \int_{\hat{M}} \langle W_t \circ g_t, Y_t \circ g_t \rangle = \int_{\hat{M}} \langle W_t \circ g_t, Y_t \circ g + t \rangle \underbrace{|\det g'_t|}_{=1} = \int_M W_t^\flat(Y_t)$$

for all $t \in [a, b]$ and $Y_t \in \mathrm{sdiff}(W)$.

Theorem 18.

$$\{\mathrm{grad}\, f \mid f \in C^\infty(M)\} \perp \mathrm{sdiff}(M) \tag{98}$$

Proof. $f \in C^\infty(M)$, $Y \in \mathrm{sdiff}(M)$. Then

$$\int_M \langle \mathrm{grad}\, f, Y \rangle = \int_M \mathrm{div}(fY) - \int_M f \underbrace{\mathrm{div}(Y)}_{=0} = \int_{\partial M} \langle fY, N \rangle = 0$$

□

Theorem 19.

$$\{\mathrm{grad}\, f \mid f \in C^\infty(M)\} = \mathrm{sdiff}(M)^\perp \tag{99}$$

We will give two proofs for the other inclusion "$\subset$".
1.Proof. We prove this in several steps.
From calculus we know: There exists $f \in C^\infty(M)$ with $W = \mathrm{grad}\, f$ if and only if for every $p, q \in M$ the integral

$$\int_\gamma W^\flat = \int_a^b \langle W \circ \gamma, \gamma' \rangle$$

where $\gamma(a) = p$, $\gamma(b) = q$ does not depend on γ.
From this follows easily:

Proposition 20. *$W \in \{\mathrm{grad}\, f \mid f \in C^\infty(M)\}$ if and only if $\int_\gamma W^\flat = 0$ for all closed curves γ.*

Proof.

- "⇒" $\int_\gamma (\mathrm{grad}\, f)^\flat = f(\gamma(b)) - f(\gamma(a))$.
- "⇐" Let $\gamma : [a, b] \to M$, $\tilde{\gamma} : [c, d] \to M$ be two curves in M with end points $\gamma(a) = \tilde{\gamma}(c) = p$ and $\gamma(b) = \tilde{\gamma}(d) = q$. Moreover we define

$$\varphi \in C^\infty([0, \frac{1}{2}], [a, b]) \quad \text{with} \quad \begin{cases} \varphi(t) = a \ \text{ for } \ t \in [0, \varepsilon] \\ \varphi(t) = b \ \text{ for } \ t \in [\frac{1}{2} - \varepsilon, \frac{1}{2}] \end{cases}$$

and

$$\tilde{\varphi} \in C^\infty([\frac{1}{2}, 1], [c, d]) \quad \text{with} \quad \begin{cases} \tilde{\varphi(t)} = d & \text{for } t \in [\frac{1}{2}, \frac{1}{2} + \varepsilon] \\ \tilde{\varphi(t)} = c & \text{for } t \in [1 - \varepsilon, 1] \end{cases}$$

Afterwards we define

$$\eta := \gamma \circ \varphi : [0, \frac{1}{2}] \to M$$
$$\tilde{\eta} := \tilde{\gamma} \circ \tilde{\varphi} : [\frac{1}{2}, 1] \to M$$

and finally

$$\hat{\gamma} : [0, 1] \to M$$
$$\hat{\gamma}(t) = \begin{cases} \eta(t) & \text{for } t \in [0, \frac{1}{2}] \\ \tilde{\eta}(t) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

Then

$$0 = \int_{\hat{\gamma}} W^\flat = \int_\eta W^\flat + \int_{\tilde{\eta}} W^\flat = \int_\gamma W^\flat - \int_{\tilde{\gamma}} W^\flat$$

□

Proposition 21. *$W \in \{\operatorname{grad} f \mid f \in C^\infty(M)\}$ if and only if $\int_\gamma W^\flat = 0$ for all smooth embeddings*

$$\gamma : \mathbb{S}^1 = \mathbb{R}/2\pi \to M$$

The proof needs the following definition.

Definition 27. *Let M be a two-dimensional manifold and let $\gamma_1 : [a, b] \to M$, $\gamma_2 : [c, d] \to M$ be two curves that intersect at $\gamma_1(p) = \gamma_2(q) = r$, then the intersection is called* transversal *if*

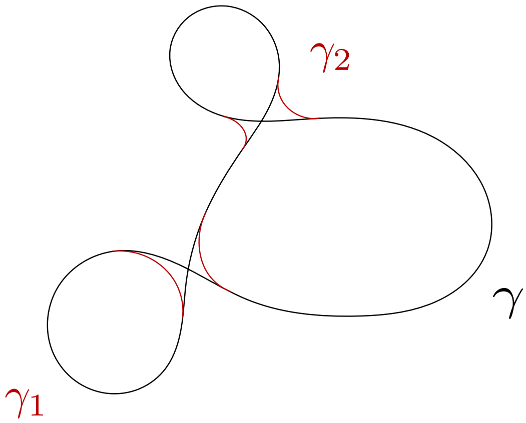
$$T_r M = d\gamma_1(\mathbb{R}) + d\gamma_2(\mathbb{R})$$

Proof.
"$\Rightarrow$" by Proposition 20.
"$\Leftarrow$" We will use two theorems from differential-topology

1. $\dim M \geq 2 \Rightarrow$ immersions $\gamma : \mathbb{S}^1 \to M$ are C^∞ dense in the space of smooth maps $C^\infty(\mathbb{S}^1, M)$.

2. $\dim M \geq 2 \Rightarrow$ small C^∞ pertubations of γ makes self-intersections transversal.

We now proceed as follows:
Small C^∞ pertubations do not change $\int_\gamma W^\flat$ much. Hence we can concentrate on immersions $\gamma : \mathbb{S}^1 \to M$ with transversal self-intersections. Then we use again statement 1.



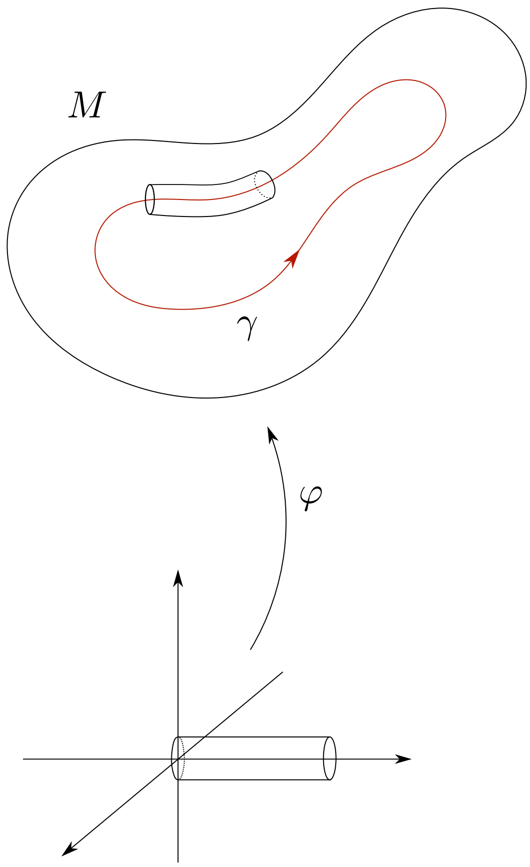
□

Now we come back to the proof of Theorem19. Due to the above considerations, it suffices to show that $\int_\gamma W^\flat = 0$ for all embeddings $\gamma : \mathbb{S}^1 = \mathbb{R}/L \to \mathring{M}$ with $|\gamma'| = 1$.
Let $\dim M = n \geq 2$ and let $(T, N_1, \ldots, N_{n-1})$ be an orthonormal frame along γ. Then

$$\varphi : \mathbb{S}^1 \times B_\varepsilon \to \mathring{M}$$
$$(t, u_1, \ldots, u_{n-1}) \mapsto \gamma(t) + \sum_{i=1}^{n-1} u_i N_i(t)$$

is a diffeomorphism for small enough ε.

-



Furthermore we define

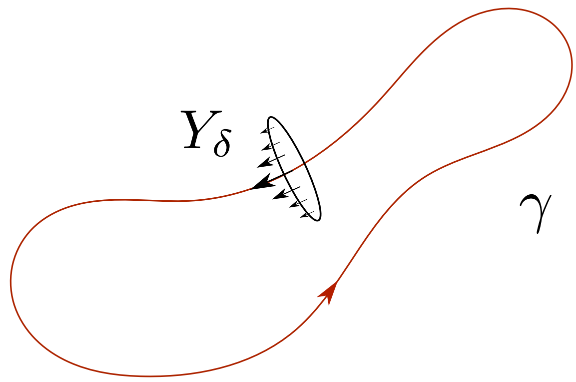
- $$r : \mathbb{S}^1 \times B_\varepsilon \to \mathbb{R}$$
  $$(t, u_1, \dots, u_{n-1}) \mapsto \sqrt{\sum_{i=1}^{n-1} u_i^2}$$

- $$\psi_\delta : (-\varepsilon, \varepsilon) \to \mathbb{R}$$
  $$\begin{cases} \psi_\delta(-x) = \psi_\delta(x) \\ \psi_\delta(x) \geq 0 \\ \psi_\delta(x) = 0 \quad \text{for} \quad |x| \geq \delta \\ \int_0^\varepsilon 2\pi \psi_\delta(x) x dx = 1 \end{cases}$$

- $$\hat{T}(\varphi(t, u_1, \dots, u_{n-1})) := T(t)$$

- $$f := \psi_\delta \circ r \circ \varphi^{-1}$$

- $$Y_q \in \mathrm{sdiff}(M)$$
  $$Y_q := \begin{cases} f \cdot T & \text{for} \quad \mathrm{q} \in \varphi(\mathbb{S}^1 \times \mathrm{B}_\varepsilon) \\ 0 & \text{otherwise} \end{cases}$$



Then

$$0 = \lim_{\delta \to 0} \int_M \langle Y_\delta, W \rangle = \int_0^L \langle \gamma', W \circ \gamma \rangle = \int_\gamma W^\flat$$

Finally we have to check: $\mathrm{div} Y = 0$.
If one define $\hat{N}_i$ just as $\hat{T}$, one can write the divergence as

$$\mathrm{div} Y = \underbrace{\langle \nabla_{\hat{T}} Y, \hat{T} \rangle}_{\mathfrak{A}} + \sum_{i=1}^{n-1} \underbrace{\langle \nabla_{\hat{N}_i} Y, \hat{N}_i \rangle}_{\mathfrak{B}}$$

- $$\mathfrak{A} = \langle (\hat{T} \cdot f)\hat{T} + f \nabla_{\hat{T}} \hat{T}, \hat{T} \rangle = 0$$

  Note that $\hat{T} \cdot f = 0$, since

  $$\hat{T} = d\varphi \left(\frac{\partial}{\partial t} \right) \quad \Rightarrow \quad df(\hat{T}) = d(\underbrace{f \circ \varphi}_{\psi \circ r}) \left(\frac{\partial}{\partial t} \right) = \frac{\partial (\psi \circ r)}{\partial t}$$

  and $\langle \nabla_{\hat{T}} \hat{T}, \hat{T} \rangle = 0$ since

  $$0 = \hat{T} \langle \hat{T}, \hat{T} \rangle = 2 \langle \nabla_{\hat{T}} \hat{T}, \hat{T} \rangle$$

- $$\mathfrak{B} = \langle (\hat{N}_i \cdot f)\hat{T} + f\nabla_{\hat{N}_i}\hat{T}, \hat{N}_i\rangle = 0$$

  $\nabla_{\hat{N}_i}\hat{T} = 0$ since

  $$\hat{N}_i = d\varphi\left(\frac{\partial}{\partial u_i}\right)$$

  is the derivative in direction u_i. But $\hat{T}$ depends on t only.

2.Proof. We start with a definition.

Definition 28. *The operator*

$$\begin{aligned} \Delta : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto \text{div grad} f \end{aligned} \tag{100}$$

is called Laplace operator*.*

We will use two theorems from the PDE theory

Theorem 22. *(Dirichlet boundary value problem)*
Let $g \in C^\infty(M)$ and $h \in C^\infty(\partial M)$ be smooth functions on M and ∂M respectively. Then there exists a unique $f \in C^\infty(M)$ such that

$$\begin{cases} \Delta f = g \\ f|_{\partial M} = h \end{cases} \tag{101}$$

Theorem 23. *(Neumann boundary value problem)*
Let $g \in C^\infty(M)$ and $h \in C^\infty(\partial M)$ be smooth functions on M and ∂M respectively. Then there exists a unique $f \in C^\infty(M)$ such that

$$\begin{cases} \Delta f = g \\ \langle \text{grad} f, N\rangle = h \end{cases} \tag{102}$$

if and only if $\int_M g = \int_{\partial M} h$. In this case f is unique up to an additive constant.

Proof. Fortunately we are in the situation that M is a subset of $\mathbb{R}^n$. So we can use the results of the classical PDE theory. For the sake of completeness we prove one direction of the Neumann boundary value problem. "$\Rightarrow$" f exists. Thus

$$\int_M g = \int_M \text{div grad} f = \int_{\partial M} \langle \text{grad} f, N\rangle = \int_{\partial M} h$$

□

Now we have all ingedients to prove the

Theorem 24. *(Helmhotz decomposition)*

$$\Gamma(TM) = \{\operatorname{grad} f \mid f \in C^\infty(M)\} \bigoplus \operatorname{sdiff}(M) \tag{103}$$

Proof.

$$\Gamma(TM) = \{\operatorname{grad} f \mid f \in C^\infty(M)\} \cap \operatorname{sdiff}(M) = \{0\}$$

follows from Theorem 18.
Conversely let $X \in \Gamma(TM)$ then

$$\int_M \underbrace{\operatorname{div} X}_{=:g} = \int_{\partial M} \underbrace{\langle X, N\rangle}_{=:h}$$

Due to Theorem 23 there exists $f \in C^\infty$ with

$$\begin{cases} \Delta f = g \\ \langle \operatorname{grad} f, N\rangle = \langle X, N\rangle \end{cases}$$

Then $Y := X - \operatorname{grad} f$ satisfies $\operatorname{div} Y = 0$, $\langle Y, N\rangle = 0$. Hence $Y \in \operatorname{sdiff}(M)$ and therefore $X = \operatorname{grad} f + Y$ □

We now finish the *2. Proof* of Theorem 19.
Let $X \in \operatorname{sdiff}(M)^\perp$, then Helmhotz tells $X = \operatorname{grad} f + Y$ with $Y \in \operatorname{sdiff}(M)$. Thus

$$0 = \int_M \langle X, Y\rangle = \underbrace{\int_M \langle \operatorname{grad} f, Y\rangle}_{=0} + \int_M \langle Y, Y\rangle$$

Hence $Y = 0$ and $X = \operatorname{grad} f$.

With Theorem 19 established we know: $t \mapsto g_t \in \tilde{\mathcal{M}}$ is critical for the action if and only if $\forall t$ there is $p_t \in C^\infty(M)$ such that $\dot{v} + \nabla_v v = -\operatorname{grad} p_t$.

How to determine p_t for given v_t?
Since $\dot{v}_t \in \operatorname{sdiff}(M)$ one obtains

$$\begin{cases} 0 = \operatorname{div}\dot{v} = -\operatorname{div}(\nabla_v v + \operatorname{grad} p_t) \\ 0 = \langle v_t, N\rangle = -\langle \nabla_v v + \operatorname{grad} p_t, N\rangle \end{cases} \Leftrightarrow \begin{cases} \Delta p_t = -\operatorname{div}(\nabla_v v) \\ \langle \operatorname{grad} p_t, N\rangle = -\langle \nabla_v v, N\rangle \end{cases} \tag{104}$$

Roughly speaking, all numerical methods proceed as follows

1. Given $v_t \in \operatorname{sdiff}(M)$
2. Compute $v_{t+\delta}$ as $v_{t+\delta} = v_t - \delta \nabla_{v_t} v_t - \operatorname{grad} q$ where q is determined by

$$\begin{cases} 0 = \operatorname{div}(\nabla_{v_t} v_t + \operatorname{grad} q) \\ 0 = \langle \delta \nabla_{v_t} v_t + \operatorname{grad} q, N\rangle \end{cases} \Leftrightarrow \begin{cases} \Delta q = -\delta \operatorname{div}(\nabla_{v_t} v_t) \\ \langle \operatorname{grad} q, N\rangle = -\delta \langle \nabla_{v_t} v_t, N\rangle \end{cases} \tag{105}$$

Solving this Neumann problem for q is called *pressure projection* of $v_t + \delta \nabla_{v_t} v_t$ to $\operatorname{sdiff}(M)$

Definition 29. *The equations*

$$\begin{cases} \dot{v} + \nabla_v v = -\operatorname{grad} p \\ \operatorname{div} \dot{v} = 0, \quad \langle \dot{v}, N\rangle = 0 \end{cases} \tag{106}$$

are called incompressible Euler equations.

4.1 Circulation

Definition 30. *Let $v \in \text{diff}_0(TM)$ be a vector field tangent to the boundary and let $\gamma : \mathbb{S}^1 \to M$ be a closed curve, then*

$$\int_\gamma v^\flat = \int_0^{2\pi} \langle v(\gamma(t)), \gamma'(t)\rangle dt \tag{107}$$

is called the circulation *of v around γ.*

We already know: There exists a function $f \in C^\infty(M)$ such that $\operatorname{grad} f = v$ ($df = v^\flat$) if and only if the circulation of v along every closed curve γ vanishes.

Example 5. *Let $M = \mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ de the unit disc. Moreover the vector field is given by*

$$v(x, y) := \begin{pmatrix} -y \\ x \end{pmatrix}$$

then the circulation of any closed curve γ is

$$\int_\gamma v^\flat = \int_0^{2\pi} \langle \begin{pmatrix} -\gamma_2 \\ \gamma_1 \end{pmatrix}, \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} \rangle dt = \int_0^{2\pi} \det(\gamma, \gamma') dt \overset{Stokes}{=} 2 \operatorname{area}(\gamma)$$

More generally, let $f : \mathbb{D}^2 \to M$ be a surface in M and let $\gamma := f|_{\partial \mathbb{D}^2}$ be its boundary. Then

$$\int_\gamma v^\flat \overset{Stokes}{=} \int_f \langle \operatorname{curl} v, N \rangle \tag{108}$$

Figuratively speaking this means: If γ bounds a piece of a surface f then the circulation of v along γ equals the flux of $\operatorname{curl} v$ through f. Thus, the circulation of v around small loops determines $\operatorname{curl} v$.

Example 6. *Let $M = \mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid r \le x^2 + y^2 \le R\}$ be an anulus and*

$$v(x, y) := \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$$

Furthermore we define the argument function $\arg : \mathbb{C} \setminus \{0\}$ *such that* $\arg$ *is smooth away from the negative real axis. Then the circulation of v around any loop γ is zero if the number of transversal intersections with the negative real axis is even. The counterexample for odd intersections is as follows*

$$r < \rho < R \qquad \gamma(t) = \begin{pmatrix} \rho \cos t \\ \rho \sin t \end{pmatrix}$$

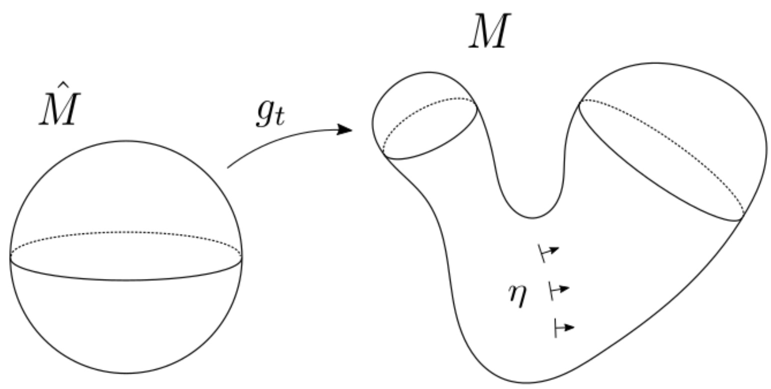
$$\int_\gamma v^\flat = \int_0^{2\pi} \langle \frac{1}{\rho^2} \begin{pmatrix} -\rho \sin t \\ \rho \cos t \end{pmatrix}, \rho \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \rangle dt = \int_0^{2\pi} 1 = 2\pi$$

So, a circulation around γ's comtain mor information then curl v.

Theorem 25. *Let $v, \tilde{v} \in \operatorname{sdiff}(M)$ be two divergence free vector fields tangent to the boundary such that $\int_\gamma v^\flat = \int_\gamma \tilde{v}^\flat$ for all loops $\gamma \in M$ then v and $\tilde{v}$ coincide.*

Proof. Since $\int_\gamma v^\flat - \tilde{v}^\flat = 0$ for all γ we have $\underbrace{v - \tilde{v}}_{\in \operatorname{sdiff}(M)} = \underbrace{\operatorname{grad} f}_{\in \operatorname{sdiff}(M)^\perp}$. Due to the Helmhotz decomposition one has $\operatorname{sdiff}(M) \cap \operatorname{sdiff}(M)^\perp = 0$ and therefore $v - \tilde{v} = 0$. □

Thus a 1-form η encodes all circulations around γ's independent of the metric (depends on the manifold structure of M only).



We now consider 1-forms under the pullback. Let $g_t : \hat{M} \to M$ then we define

$$\begin{aligned} \hat{\eta}_t &:= g_t^*(v_t^\flat) \\ \hat{\eta}_t(X) &= \langle v_t \circ g_t, dg_t(X) \rangle \quad \forall X \in \Gamma(T\hat{M}) \end{aligned} \tag{109}$$

Recall

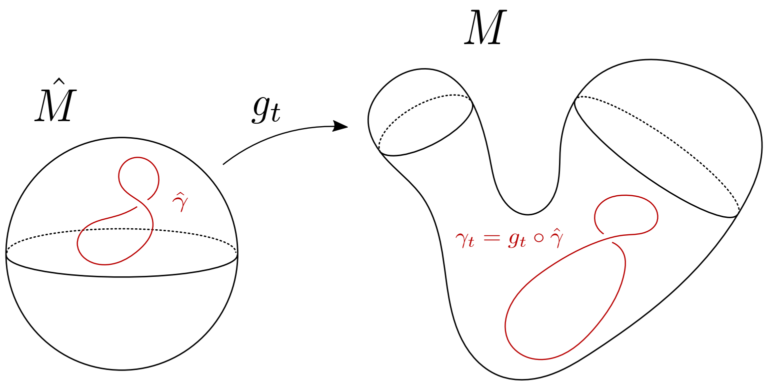
$$\begin{aligned} \dot{g}_t &= v_t \circ g_t \\ \ddot{g}_t &= (\dot{v}_t + \nabla_{v_t} v_t) \circ g_t \overset{Euler}{=} -\operatorname{grad} p_t \circ g_t \end{aligned}$$

Now the time derivative reads

$$\begin{aligned} \dot{\hat{\eta}}_t(X) &= \underbrace{\langle \operatorname{grad} p_t \circ g_t, dg_t(X) \rangle}_{-d(p_t \circ g_t)(X)} + \underbrace{\langle v_t \circ g_t, \underbrace{(dg_t(x))^\cdot}_{d(v_t \circ g_t)(X)} \rangle}_{\frac{1}{2} d(\underbrace{\langle v_t \circ g_t, v_t \circ g_t \rangle}_{\langle v_t, v_t \rangle \circ g_t})(X)} \\ &= \underbrace{d((\underbrace{\frac{1}{2}|v_t|^2 - p_t}_{=:-\beta_t}) \circ g_t)}_{-\hat{\beta}_t} \end{aligned}$$

Thus

$$\dot{\hat{\eta}}_t = -d\hat{\beta}_t, \quad \hat{\beta}_t \in C^\infty(\hat{M}) \tag{110}$$



and

$$\int_{\gamma_t} \eta_t = \int_0^{2\pi} \eta_t(\underbrace{\gamma_t'}_{dg_t(\hat{\gamma}')}) = \int_0^{2\pi} \hat{\eta}_t(\hat{\gamma}') = \int_{\hat{\gamma}} \hat{\eta}_t \tag{111}$$

Theorem 26. (Kelvin - Helmhotz circulation theorem)
Let $t \mapsto v_t \in \operatorname{diff}_0(M)$ corresponding to the notation of a barotropic fluid (i.e $\dot{v}_t + \nabla_{v_t} v_t = -\operatorname{grad} p_t$ with $p_t \in C^\infty(M)$) then for every loop $\hat{\gamma} \in M$ there is a constant $c \in \mathbb{R}$ such that for $\gamma_t = g_t \circ \hat{\gamma}$ and $\eta_t = v_t^\flat$ we have

$$\int_{\gamma_t} \eta_t = c$$

Proof.

$$\left(\int_{\gamma_t} \eta_t\right)^{\cdot} \overset{(111)}{=} \int_{\hat{\gamma}} \dot{\hat{\eta}} = -\int_{\hat{\gamma}} d\alpha = 0$$

□

4.2 Lie Derivative

Let $g : \hat{M} \to M$ be a diffeomorphism. Then every object on $\hat{M}$ can be transported via g to M

- $f \in C^\infty(\hat{M})$

$$T_g f \in C^\infty(M)$$
$$(T_g f)(q) := f(g^{-1}(q))$$
$$T_g f := f \circ g^{-1} = (g^{-1})^* f \tag{112}$$

- $X \in \Gamma(T\hat{M})$

$$\begin{aligned} T_g X &\in \Gamma(TM) \\ (T_g X)_q &= dg(X_{g^{-1}(q)}) \\ T_g X &= g_* X \end{aligned} \tag{113}$$

- $\omega \in \Omega^k(\hat{M})$ and $X_1, \dots X_k \in T_q M$

$$\begin{aligned} T_g \omega &\in \Omega^k(M) \\ T_g \omega(X_1, \dots X_k) &= \omega_{g^{-1}}(dg^{-1}(X_1), \dots, dg^{-1}(X_k)) \\ T_g \omega &= (g^{-1})^* \omega \end{aligned} \tag{114}$$

T_g plays together nicely with all natural operations.

- $T_g(d\omega) = d(T_g\omega)$, $T_g([X, Y]) = [T_g X, T_g Y]$
- $T(X \cdot f) = (T_g X) \cdot (T_g f)$
- $T_g(\omega(X_1, \dots, X_k)) = T_g\omega(T_g X_1, \dots, T_g X_k)$

Definition 31. *Let $X \in \Gamma(TM)$ a smooth vector field on a compact manifold M with boundary and let $g_t : M \to M$ be a diffeomorphism for $t \in \mathbb{R}$ satisfying the initial value problem*

$$\begin{cases} \frac{d}{dt} g_t(p) = X_{g_t(p)} \\ g_0 = id \end{cases} \tag{115}$$

then the Fisherman's derivative $\mathcal{F}$ *is given by*

- *for* $\omega \in \Omega^k(M)$

$$(\mathcal{F}_X \omega) q := \frac{d}{dt}\Big|_{t=0} (T_{g_t} \omega)_q \tag{116}$$

and similarly

- *for* $Y \in \Gamma(TM)$

$$(\mathcal{F}_X Y) q := \frac{d}{dt}\Big|_{t=0} (T_{g_t} Y)_q \tag{117}$$

Properties

- $\mathcal{F}_X(d\omega) = d(\mathcal{F}_X \omega)$
- $\mathcal{F}_X([Y, Z]) = [\mathcal{F}_X Y, Z] + [Y, \mathcal{F}_X Z]$
- $\mathcal{F}_X(\omega(Y_1, \dots, Y_k)) = (\mathcal{F}_X \omega)(Y_1, \dots, Y_k) + \omega(\mathcal{F}_X Y_1, \dots, Y_k) + \dots + \omega(Y_1, \dots, \mathcal{F}_X Y_k)$
- $(\mathcal{F}_X f)_q = df(-X_q) = -(Xf)_q$

Definition 32. *Let $\mathcal{F}$ be the Fisherman's derivative, then the* Lie derivative $\mathcal{L}$ *is given by*

$$\mathcal{L}_X \omega := -\mathcal{F}_X \omega \tag{118}$$

$$\mathcal{L}_X Y := -\mathcal{F}_X Y \tag{119}$$

and

$$\mathcal{L}_X f = Xf \tag{120}$$

Analogously to the Fisherman's derivative one obtains the following properties

- $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$
- $\mathcal{L}_X(\omega(Y_1, \ldots, Y_k)) = (\mathcal{L}_X \omega)(Y_1, \ldots, Y_k) + \omega(\mathcal{L}_X Y_1, \ldots, Y_k) + \cdots + \omega(Y_1, \ldots, \mathcal{L}_X Y_k)$

Lemma 1.

$$\mathcal{L}_X Y = [X, Y] \tag{121}$$

Proof.

$$X(Yf) = \mathcal{L}_X(Yf) = (\mathcal{L}_X Y) \cdot f + \underbrace{Y \cdot \mathcal{L}_X f}_{Y(Xf)}$$

□

Definition 33. *Let $\omega \in \Omega^k(M)$ be a k-form on M, then for $X \in \Gamma(TM)$ the map*

$$\begin{aligned} \iota_X &: \Omega^k(M) \to \Omega^{k-1}(M) \\ \iota_X \omega(Y_2, \ldots, Y_k) &:= \omega(X, Y_2, \ldots, Y_k) \end{aligned} \tag{122}$$

is called interior derivative

Lemma 2. *Let $\omega \in \Omega^1(M)$ be a 1-form on M, then for $X \in \Gamma(TM)$ holds*

$$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega \tag{123}$$

Proof. Take $Y \in \Gamma(TM)$, then

$$X\omega(Y) = \mathcal{L}_X(\omega(Y)) = (\mathcal{L}_X \omega)(Y) + \omega([X, Y])$$

and

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

hence

$$(\mathcal{L}_X \omega)(Y) = d\omega(X, Y) + Y\omega(X) = \iota_X d\omega(Y) + d(\iota_X \omega)(Y)$$

□

Theorem 27. *(Cartans magic formula)*
Let $\omega \in \Omega^k(M)$ be a k-form on M and let $X \in \Gamma(TM)$, then

$$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega \tag{124}$$

Proof. by induction over k. Use Lemma 2 as initial step. □

We already know: g_t is volume preserving (i.e $T_{g_t} \det = \det$) if and only if $\operatorname{div} X = 0$. The next theorem is a reformulation in terms of Lie derivative.

Theorem 28. *Let $g_t : M \to M$ be a diffeomorphism, then g_t is volume preserving if and only if $\mathcal{L}_X \det = 0$.*

Proof. Firstly we prove

$$d(\iota_X \det) = (\operatorname{div} X) \det \tag{125}$$

Let $p \in M$, choose an orthonormal frame field $Y_1, \dots Y_n$ near p with $(\nabla Y_j)_p = 0$ and express X as

$$X := \sum_{i=1}^{n} u_i Y_i$$

then

$$\begin{aligned} d(\iota_X \det)(Y_1, \dots, Y_k)_p &= \sum_{j=1}^{n} (-1)^{j+1} Y_j (\iota_X \det)(Y_1, \dots, \hat{Y}_j, \dots, Y_n) \\ &= \sum_{j=1}^{n} Y_j (\det(Y_1, \dots, \underset{\substack{\uparrow \\ j}}{X}, \dots, Y_n)) \end{aligned}$$

and

$$\begin{aligned} \operatorname{div} X_p = \operatorname{tr}(\nabla X)_p &= \sum_{j=1}^{n} \langle \nabla_{Y_j} X, Y_j \rangle \\ = \sum_{j=1}^{n} \langle \nabla_{Y_j} \sum_{i=1}^{n} u_i Y_i, Y_j \rangle &= \sum_{j=1}^{n} \langle \sum_{i=1}^{n} (Y_j u_i) Y_i, Y_j \rangle \end{aligned}$$

Finally we use Cartan's magic formula

$$\mathcal{L}_X \det = \iota_X \underbrace{d \det}_{(n+1) \text{ form}=0} + d\iota_X \det$$

□

This theorem can be seen as a non Riemannian definition of divergence.
We now derive the Euler equations in terms of the Lie derivative of the velocity 1-form $\eta = v^\flat$.

$$\begin{aligned}(\mathcal{L}_{\eta^\sharp}\eta)(X) &= \mathcal{L}_{\eta^\sharp}(\eta(X)) - \eta(\mathcal{L}_{\eta^\sharp}X) \\ &= v \cdot \langle v, X\rangle - \langle v, [v, X]\rangle \\ &= \langle \nabla_v v, X\rangle + v, \nabla_v X\rangle - \langle v, \nabla_v X - \nabla_X v\rangle \\ &= \langle \nabla_v v, X\rangle + \frac{1}{2} X \cdot \langle v, v\rangle \\ &= \langle \nabla_v v + \frac{1}{2}\operatorname{grad}|v|^2, X\rangle\end{aligned}$$

hence

$$(\mathcal{L}_{\eta^\sharp}\eta)^\sharp = \nabla_v v + \frac{1}{2}\operatorname{grad}|v|^2 \quad \Leftrightarrow \quad \nabla_v v = (\mathcal{L}_{\eta^\sharp}\eta)^\sharp - \frac{1}{2}\operatorname{grad}|v|^2$$

plugging in the equation of motion $\dot{v} + \nabla_v v = -\operatorname{grad} p$ yields

$$\dot{v} + (\mathcal{L}_{\eta^\sharp}\eta)^\sharp = -\operatorname{grad}(\underbrace{p - \frac{1}{2}|v|^2}_{=:\beta})$$

Applying the $\flat$ operator and using $(\operatorname{grad} f)^\flat = df$ gives us the Euler equation

$$\dot{\eta} + \mathcal{L}_{\eta^\sharp}\eta = -d\beta \tag{126}$$

4.3 Vorticity

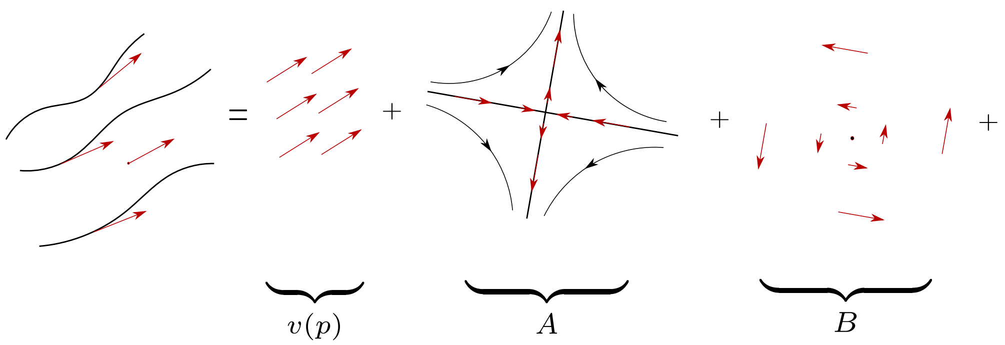
Let $v \in \Gamma(TM)$ be a vector field. At each $p \in M$ decompose

$$\nabla v : X \mapsto \nabla_X v \tag{127}$$

into its symmetric and skew parts

$$\nabla v = \frac{1}{2}\underbrace{(\nabla v + (\nabla v)^*)}_{=:A} + \frac{1}{2}\underbrace{(\nabla v - (\nabla v)^*)}_{=:B} \tag{128}$$

Consider the taylor expansion of the vector field around p



Theorem 29. *Let $\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$ denote the metric on M then*

$$\langle \nabla_X v, Y\rangle = \frac{1}{2}(\mathcal{L}_v g + d\eta)(X,Y) \tag{129}$$

Proof.

$$\begin{aligned}(\mathcal{L}_v g)(X,Y) &= \mathcal{L}_v(g(X,Y)) - g(\mathcal{L}_v X, Y) - g(X, \mathcal{L}_v Y)\\ &= \underbrace{v\cdot\langle X,Y\rangle}_{\langle \nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle} - \langle \nabla_v X - \nabla_X v, Y\rangle - \langle X, \nabla_v Y - \nabla_Y v\rangle\\ &= \langle \underbrace{\nabla_X v}_{(\nabla v)(X)}, Y\rangle + \underbrace{\langle X, \underbrace{\nabla_Y v}_{(\nabla v)(Y)}\rangle}_{\langle (\nabla v)^* X, Y\rangle}\\ &= \langle (\nabla_v + (\nabla_v)^*)X, Y\rangle = \langle 2AX, Y\rangle\end{aligned}$$

In addition, this shows that the Lie derivative of a metric is still a symmetric tensor.

We now look at the exterior derivative of the velocity 1-form $\eta = v^\flat$.

$$\begin{aligned}d\eta(X,Y) &= X\langle v, Y\rangle - Y\langle v, X\rangle - \langle v, \nabla_X Y - \nabla_Y X\rangle\\ &= \langle \nabla_X v, Y\rangle - \langle \nabla_Y v, X\rangle\\ &= \langle (\nabla_v - (\nabla_v)^*)X, Y\rangle = \langle 2BX, Y\rangle\end{aligned}$$

□

At this point we investigate the special case $\dim M = 3$.
Fore each $\omega \in \Omega^2(M)$ there is a unique $\mathbf{w} \in \Gamma(TM)$ such that

$$\begin{aligned}\omega &= \iota_{\mathbf{w}} \det\\ \omega(X,Y) :&= \det(\mathbf{w}, X, Y) =: \langle \mathbf{w}, X \times Y\rangle\end{aligned} \tag{130}$$

In particular for $\omega = d\eta$

$$\nabla_X v = \frac{1}{2}(\iota_X \mathcal{L}_v g)^\sharp + \frac{1}{2}\mathbf{w} \times X \tag{131}$$

Proposition 30. *For* $\dim M = 3$ *and* $d\eta = \iota_{\boldsymbol{w}} \det$

$$v \times \boldsymbol{w} = \frac{1}{2}\operatorname{grad}|v|^2 - \nabla_v v \tag{132}$$

Proof.

$$\begin{aligned}\langle v \times \mathbf{w}, Y\rangle &= \langle \mathbf{w}, Y \times v\rangle = d\eta(Y \times v)\\ &= Y \underbrace{\eta(v)}_{\langle v,v\rangle} - \underbrace{\underbrace{v\eta(Y)}_{v\langle v, Y\rangle}}_{\langle \nabla_v v, Y\rangle + \langle v, \nabla_v Y\rangle} - \underbrace{\eta([Y,v])}_{\langle v, \nabla_Y v - \nabla_v Y\rangle}\\ &= Y\langle v,v\rangle - \langle \nabla_v v, Y\rangle - \underbrace{\langle v, \nabla_Y v\rangle}_{\frac{1}{2}Y\langle v,v\rangle}\\ &= \langle \frac{1}{2}\operatorname{grad}|v|^2 - \nabla_v v, Y\rangle\end{aligned}$$

□

Let's come back to the equations ot motion $\dot{v} + \nabla_v v = -\operatorname{grad} p$. Plugging in equation (132) leads to

$$\dot{v} + \mathbf{w} \times v = -\operatorname{grad}(\underbrace{p + \frac{1}{2}|v|^2}_{=:\alpha}) \tag{133}$$

Definition 34. *α is called the* Bernoulli funtion *of v.*

We have three kinds of equations of motion

$$\dot{v} + \begin{pmatrix} \mathbf{w} \times v \\ \nabla_v v \\ (\mathcal{L}_v \eta)^\sharp \end{pmatrix} = -\operatorname{grad} \begin{pmatrix} p + \frac{1}{2}|v|^2 \\ p \\ p - \frac{1}{2}|v|^2 \end{pmatrix} \tag{134}$$

4.4 Stationary Flows

Before we proceed we racall some facts from calculus

Definition 35. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^k$ be a smooth function, then $q \in \mathbb{R}^k$ is called a* critical point *of f if there is $p \in U$ with $f(p) = q$ and* $\operatorname{rank}(f'(g)) < k$ *(*regular value *otherwise).*

Lemma 3. *(Lemma of Sarde)*
The set of critical values of f has measure zero.

Note $q \notin f(U)$ then q is a regular value.

Example 7. $k = 1$*: Then $q \in \mathbb{R}$ is a regular value if there is no $p \in U$ such that $f(p) = q$ and $f'(p) = 0$.*

The preimage of a regular value $q \in \mathbb{R}^k$ is either empty or a smooth submanifold of U.

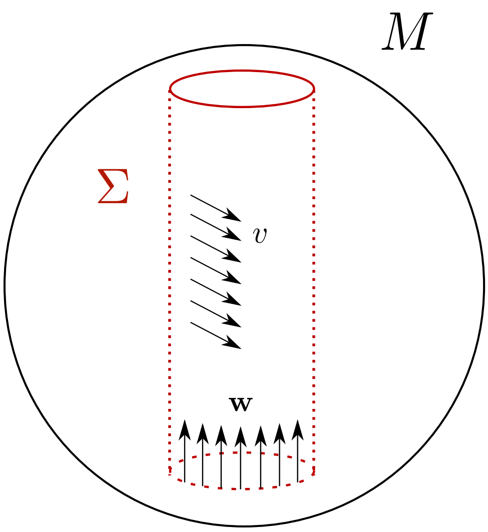
Definition 36. *A flow is called* stationary *if $\dot{v} = 0$.*

Then the equation of motion becomes

$$v \times \mathbf{w} = \operatorname{grad} \alpha$$

Case 1: α is not a constant function.
Let $q \in \mathbb{R}$ be a regular value of α then $\Sigma := \alpha^{-1}\{q\} \neq \emptyset$ and hence Σ is a smooth, orientable, compact surface possibly with boundary $\partial\Sigma = \Sigma \cap \partial M$.

- $v|_\Sigma$, $\mathbf{w}|_\Sigma$ are tangent to Σ since $\operatorname{grad} \alpha$ points in the normal direction of the levelset.
- $v|_\Sigma$, $\mathbf{w}|_\Sigma$ are linearly independent since q was assumed to be a regular value.



One can say for $v \in \operatorname{sdiff}(M)$ stationary: $v_t = v$ is a solution of equations of motion if and only if there exists $\alpha \in C^\infty(M)$ such that $v \times \operatorname{curl} v = \operatorname{grad} \alpha$.

Example 8.

$$M := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\} \qquad v(x, y, z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

then

$$\operatorname{curl} v(x, y, z) = \begin{pmatrix} 0 \\ 0 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

and

$$(v \times \boldsymbol{w})(x, y, z) = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} = \operatorname{grad}(\alpha(x, y, z))$$

where $\alpha(x, y, z) = x^2 + y^2$.

Theorem 31. *(proof later)*
Let Σ *be a compact oriented surface with boundary which admits a nowhere vanishing vector field* $v \in \operatorname{diff}(M)$*. Then every connected component of* Σ *is diffeomorph to* $[0, 1] \times \mathbb{S}^1$ *or* $\mathbb{S}^1 \times \mathbb{S}^1$*.*

What is about the motion on Σ?
Applying the exterior derivative to the vorticity equation $\dot{\eta} + \mathcal{L}_v \eta = -d\beta$ and setting $\omega = d\eta$ leads to

$$\dot{\omega} + \mathcal{L}_v \omega = 0 \tag{135}$$

Using the chain rule yields

$$(\mathcal{L}_v\omega)(X,Y) = \mathcal{L}_v(\omega(X,Y)) - \omega(\mathcal{L}_vX,Y) - \omega(X,\mathcal{L}_vY)$$

Since ω is defined as $\omega = \iota_{\mathbf{w}}\det$ and $\mathcal{L}_v\det = 0$ because it is divergence free, one obtains

$$\begin{aligned}\mathcal{L}_v(\omega(X,Y)) &= \mathcal{L}_v(\det(\mathbf{w},X,Y))\\ &= \underbrace{(\mathcal{L}_v\det)}_{=0} + \det([v,\mathbf{w}],X,Y) + \det(\mathbf{w},[v,X],Y) + \det(\mathbf{w},X,[v,Y])\end{aligned}$$

Hence

$$\dot{\omega} = (\iota_{\mathbf{w}}\det)^{\cdot} = -\iota_{[v,\mathbf{w}]}\det$$

Since the determinant depends not on time

$$\iota_{\dot{\mathbf{w}}}\det = -\iota_{[v,\mathbf{w}]}\det$$

Therefore the vorticity equation becomes

$$\dot{\mathbf{w}} = [v,\mathbf{w}] \tag{136}$$

Theorem 32. *If $t \mapsto v_t \in \mathrm{sdiff}(M)$ satisfying the Euler equations, then the vorticity vector field $t \mapsto \boldsymbol{w}$ is frozen into the fluid in the sense that there is a time independent vector field $\widehat{\boldsymbol{w}} \in \Gamma(T\hat{M})$ such that*

$$\boldsymbol{w}_t = g_{t*}\widehat{\boldsymbol{w}} \tag{137}$$

Proof. Since incompressible fluids are volume preserving we have $T_{g_t}\widehat{\det} = \det$ and therefore

$$T_{g_t}\iota_{\widehat{\mathbf{w}}_t}\widehat{\det} = \iota_{\mathbf{w}_t}\det$$

Using the definitions of ω and η one obtains

$$\iota_{\widehat{\mathbf{w}}_t}\widehat{\det} = d\hat{\eta}_t$$

Recall identity (110)

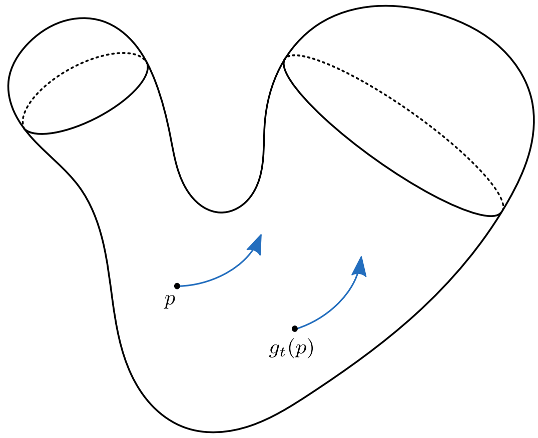
$$\dot{\hat{\eta}}_t = -d\hat{\beta}_t$$

Hence

$$d\dot{\hat{\eta}}_t = 0 \quad \Rightarrow \quad \dot{\widehat{\mathbf{w}}}_t = 0$$

□

Figuratively speaking, at time 0 draw a curved arrow along a vortex line with blue ink, wait a time t, look what became of the blue arrow.



Then then the new evolved arrow is a vortex line at time t.

Example 9. *Construct* $\boldsymbol{w}$ *to be non-zere only inside a certain knotted tube. Moreover let* $v \in \mathrm{sdiff}(M)$ *with* $\operatorname{curl} v = \boldsymbol{w}$*. Then after evolving* v *the vorticity* $\boldsymbol{w}_t$ *is concentrated in the knotted tube for all time.*

Note that the vorticity equation for a stationary flow is

$$[v, \mathbf{w}] = 0$$

i.e. v and $\mathbf{w}$ commute.

Theorem 33. *Let* M *be a compact manifold with boundary and* $X, Y \in \mathrm{sdiff}(M)$ *with* $[X, Y] = 0$*. Moreover let* $g_t, h_t \in \mathrm{Diff}_0(M)$ *be solutions of the initial value problems*

$$\begin{cases} \frac{\partial g_t}{\partial t} = X \circ g_t \\ g_0 = id_M \end{cases} \qquad \begin{cases} \frac{\partial h_s}{\partial s} = Y \circ h_s \\ h_0 = id_M \end{cases}$$

then for all $s, t \in \mathbb{R}$

$$h_s \circ g_t = g_t \circ h_s$$

Proof. Fix $t \in \mathbb{R}$ and define $\tilde{h}_s \in \mathrm{Diff}_0(M)$ piontwise by

$$\tilde{h}_s := g_t \circ h_s(p) \circ g_t^{-1}(p)$$

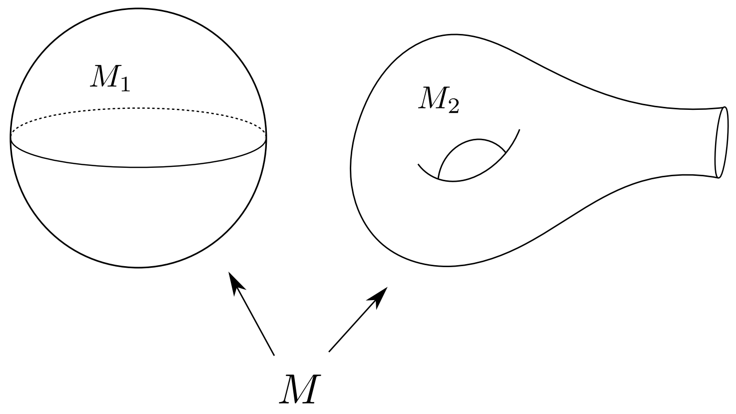
Then the corresponding vector field is $\tilde{Y} = T_{g_t} Y = g_{t*} Y$. The fact that

$$\tilde{Y} = Y \quad \Leftrightarrow \quad \mathcal{L}_X Y = [X, Y] = 0$$

proves the statement. $\square$

5 Cohomology of Compact Manifolds with Boundary

Let M be a compact oriented n-dimensional manifold with boundary, $M = M_1 \cup \cdots \cup M_m$ where $M_1, \ldots, M_m$ are the connected components of M.



$$\begin{array}{ccccccccc} \Omega^{-1}(M) & \xrightarrow{d_{-1}} & \Omega^0(M) & \xrightarrow{d_0} & \Omega^1(M) & \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} & \Omega^n(M) & \xrightarrow{d_n} & \Omega^{n+1}(M) \\ \| & & \| & & & & & & \| \\ \{0\} & & C^\infty(M) & & & & & & \{0\} \end{array}$$

Definition 37. *Let $\omega \in \Omega^k(M)$ be a k-form. Then ω*

- *is called* closed *if $\omega \in \operatorname{Ker} d_k$*
- *is called* exact *if $\omega \in \operatorname{Im} d_{k-1}$*

Since $d_k \circ d_{k-1} = 0 \quad \Rightarrow \quad \operatorname{Im} d_{k-1} \subset \operatorname{Ker} d_k$.

Definition 38.

$$H^k(M) := \operatorname{Ker} d_k \Big/ \operatorname{Im} d_{k-1} \tag{138}$$

is called the k-th cohomology.

Theorem 34. *$H^k = \{0\}$ for $k < 0$ and $k > n$. $H^0(M), \ldots, H^n(M)$ are all finite dimensional.*

Definition 39. *$\beta_k(M) := \dim H^k(M)$ is called the k-th* Betti number *of M.*

Example 10. *Consider* $\operatorname{Ker} d_0 = \{f \in C^\infty(M) \mid df = 0\}$ *then*

$$f \in \operatorname{Ker} d_0 \Leftrightarrow f|_{M_j} = const \;\Rightarrow\; H^0(M) \simeq \mathbb{R}^m \text{ and } \beta_0(M) = m$$

since $\operatorname{Im} d_{-1} = \{0\}$.
$\operatorname{Ker} d_n = \Omega^n(M)$ *because all* n*-forms are closed.*

Suppose $\partial M = \emptyset$. Due to Stokes theorem

$$\int_M d\eta = 0 \qquad \forall \eta \in \Omega^{n-1}(M)$$

So, if $\omega \in \operatorname{Im} d_{d-1} \;\Rightarrow\; \int_M \omega = 0$. On the other hand, if ω is a volume form $\int_M \omega = \operatorname{vol}(\omega) > 0$. Hence the volume form ω is not exaxt.

Theorem 35. ***(Ditichlet Problem, manifold version)***
Let M *be a compact connected Riemannian manifold with boundary and* $\partial M \neq \emptyset$, $H \in C^\infty(M)$, $g \in C^\infty(M)$. *Then there is a unique* $f \in C^\infty(M)$ *with*

$$\begin{cases} \Delta f = g \\ f|_{\partial M} = h \end{cases}$$

Theorem 36. *Let* M *be a compact connected oriented* n*-dim manifold with* $\partial M \neq \emptyset$. *Then* $H^n(M) = \{0\}$. *"All* n*-forms are exact".*

Proof. Choose a Riemannian metric on M. Rewrite $\omega \in \Omega^n$ as

$$\omega = g \cdot \det \quad \text{with} \quad g \in C^\infty(M)$$

where $\det$ denotes the volume form. Due to the Dirichlet theorem there exists $f \in C^\infty(M)$ with $\Delta f = g$, $f|_{\partial M} = 0$. Then

$$\begin{aligned} & \operatorname{div} \operatorname{grad} f = g \\ \Leftrightarrow \;& \operatorname{div} \operatorname{grad} f \cdot \det = g \cdot \det \\ \overset{(125)}{\Leftrightarrow} \;& d(\iota_{\operatorname{grad} f} \det) = g \cdot \det = \omega \end{aligned}$$

□

Theorem 37. ***(Neuman Problem, manifold version)***
Let M *be a compact connected Riemannian manifold with boundary. Moreover let* $g \in C^\infty(M)$ *and* $h \in C^\infty(\partial M)$ *be smooth functions on* M *and* ∂M *respectively satisfying* $\int_M g = \int_{\partial M} h$. *Then there is a* $f \in C^\infty(M)$ *(unique up to an additive constant) such that*

$$\begin{cases} \Delta f = g \\ df(N) = h \end{cases}$$

where N *is the outward pointing unit normal along* ∂M.

Remark 2. *If* $\int_M g \neq \int_{\partial M} h$ *then*

$$\begin{cases} \Delta f = g \\ df(N) = h \end{cases}$$

has no solution f.

Proof. Using the Divergence theorem yields

$$\int_M \Delta f = \int_M \operatorname{div} \operatorname{grad} f = \int_{\partial M} \langle \operatorname{grad} f, N \rangle = \int_{\partial M} df(N)$$

□

Theorem 38. *Let* M *be a compact connected oriented manifold without boundary. Then* $\dim H^n(M) = 1$.

Proof. By Stokes $\int_M \eta = 0$ for all $\eta \in \operatorname{Im} d_{n-1}$. So if $\omega \in \Omega^n(M)$ then $\int_M \omega + \eta = \int_M \omega$. Hence $\int_M \omega$ depends on the cohomology class of ω only

$$[\omega] \in H^n(M) = \Omega^n(M) \big/ \operatorname{Im} d_{n-1} \tag{139}$$

So $\int_M [\omega] := \int_M \omega$ is well defined.
Finally we prove that the linear map

$$\int_M : H^n(M) \to \mathbb{R} \tag{140}$$

is bijective.

- **surjective**: set $\omega = c \cdot \det$ with $c \in \mathbb{R}$
- **injective**: Choose a Riemannian metric, $[\omega] \in \operatorname{Ker} \int_M$ i.e. $[\omega] \in H^n(M)$ with $\int_M \omega = 0$. Set $\omega = g \cdot \det$ hence $\int_M \omega = 0$. By Neumann $\exists f \in C^\infty(M)$ with $\Delta f = g$. Therefore

$$\begin{aligned} &\operatorname{div}(\operatorname{grad} f) = g \\ \overset{(125)}{\Leftrightarrow} \quad & d(\iota_{\operatorname{grad} f} \det) = g \cdot \det = \omega \\ &\Rightarrow \omega \in \operatorname{Im} d_{n-1} \Rightarrow [\omega] = 0 \end{aligned}$$

  Since (140) is linear this proves the injectivity.

□

Easy to see: $M_1, \dots, M_m$ compact oriented manifolds with boundary and $M = M_1 \coprod \cdots \coprod M_m$ is the disjoint union. Then $H^k(M) = H^k(M_1) \bigoplus \cdots \bigoplus H^k(M_m)$.

Corollary 1. *Let* M *be a compact oriented manifold with boundary,* $M_1, \dots, M_m$ *connected components of* M. *Then*

$$\dim H^n(M) = \#\{j \mid \partial M = \emptyset\} \tag{141}$$

Corollary 2. *Let M be a compact oriented manifold without boundary. Then*

$$\dim H^0(M) = \dim H^n(M) \tag{142}$$

This is a special case of the Poincaré duality theorem.
In order to state the theorem in its full generality we need a few definitions.

Definition 40. *Let V, W be vector spaces. Then $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{R}$ bilinear yields a linear map*

$$\begin{aligned} f : V &\to W^* \\ (f(v))(w) &:= \langle v, w \rangle \end{aligned} \tag{143}$$

$\langle \cdot, \cdot \rangle$ *is called* non-degenerate *if*

- $\langle v, w \rangle = 0$ *for all* $w \Rightarrow v = 0$
- $\langle v, w \rangle = 0$ *for all* $v \Rightarrow w = 0$

Note that, $\langle \cdot, \cdot \rangle$ is non-degenerate if and only if f is an isomorphism.

Theorem 39. ***(Poincaré duality theorem)***
Let M be a compact oriented manifold without boundary. Then

$$\begin{aligned} H^k(M) \times H^{n-k}(M) &\to \mathbb{R} \\ (\omega, \eta) \mapsto \langle \omega, \eta \rangle &= \int_M \omega \wedge \eta \end{aligned} \tag{144}$$

is well defined and it is a non-degenerate pairing between $H^k(M)$ and $H^{n-k}(M)$. So it establishes an isomorphism $H^k \leftrightarrow H^{n-k}(M)^*$. In particular $\dim H^k = \dim H^{n-k}$.

Remark 3. *Let $\beta \in \Omega^{k-1}(M)$ then*

$$\begin{aligned} \int_M (\omega + d\beta) \wedge \eta &= \int_M \omega \wedge \eta \int_M d\beta \wedge \eta \\ &\overset{d\eta=0}{=} \int_M \omega \wedge \eta \int_M d(\beta \wedge \eta) \\ &= \int_M \omega \wedge \eta \end{aligned}$$

Similarly for $\alpha \in \Omega^{n-k-1}(M)$

$$\int_M \omega \wedge (\eta + d\alpha) = \int_M \omega \wedge \eta$$

Definition 41.

$$H_k(M) := H^k(M)^* \tag{145}$$

is called the k'th homology *vector space of M.*

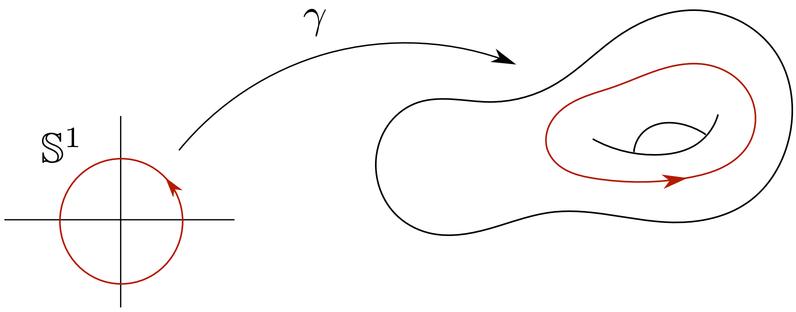
Let Σ be a compact oriented manifold, $\dim \Sigma = k$ and $\partial\Sigma = \emptyset$. Moreover let $f : \Sigma \to M$ be a smooth map. Then define $\hat{f} \in H_k(M)$ by

$$\hat{f}([\omega]) := \int_\Sigma f^*\omega \tag{146}$$

where $\omega \in \Omega^k(M)$, $d\omega = 0$. This is well defined because

$$\int_\Sigma f^*(\omega + d\eta) = \int_\Sigma f^*\omega + \int_\Sigma \underbrace{f^*d\eta}_{df^*\eta} = \int_\Sigma f^*\omega$$

Example 11. $\Sigma = S^1$



$\omega \in \Omega^1(M)$

$$\hat{\gamma}[\omega] = \int_{\mathbb{S}^1} \gamma^*\omega = \int_\gamma \omega$$

Definition 42. *$f, \tilde{f} : \Sigma \to M$ are called* homologous *if $\hat{f} = \hat{\tilde{f}}$ i.e.*

$$\int_\Sigma f^*\omega = \int_\Sigma \tilde{f}^*\omega \quad \forall\omega \in \Omega^k(M) \quad \text{with} \quad d\omega = 0$$

If you compare with books, They would call our cohomology "de Rham-cohomology". Only one book does everything.
Bott & Tu "Differetial Forms in Algebraic Topology"
See also
Milnor "Topology from the Differentiable Viewpoint"

Example 12. *Let $M := \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2+y^2 \leq 2\}$ be an anulus. Then $H^1(M) \neq \{0\}$ because $H_1(M) \neq \{0\}$. To see this choose*

$$\omega := \frac{-ydx + xdy}{x^2 + y^2}$$

then

$$
\begin{aligned}
d\omega &= -\frac{2xdx + 2ydy}{(x^2+y^2)^2} \wedge (-ydx + xdy) + \frac{-dy \wedge dx + dx \wedge dy}{x^2+y^2} \\
&= \left[2\frac{-x^2-y^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} \right] dx \wedge dy = 0
\end{aligned}
$$

We now integrate over $\gamma(t) := \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$
\int_\gamma \omega = \int_0^{2\pi} \sin^2 t + \cos^2 t \; dt = 2\pi
$$

Thus γ *is not homologous to zero, i.e.* $\hat{\gamma} \neq 0$ *because* $\hat{\gamma}([\omega]) \neq 0$.

To prepare for the next theorem we recall some basic facts from linear algebra.

- Let V be a vector space and $U \subset V$ be a linear subspace. Then

$$
U^\circ := \{ f \in V^* \mid f(u) = 0 \;\; \forall u \in U \} \tag{147}
$$

  is called the *annihilator* of U.

- In finite dimensional vector spaces the dimension of U° is given by

$$
\dim U^\circ = \dim V - \dim U \tag{148}
$$

- Thus we can conclude that $U = V$ if $U^\circ = \{0\}$.

Theorem 40. $H_1(M)$ *is spanned by* $\{\hat{\gamma} \mid \gamma : \mathbb{S}^1 \to M \;\; \text{smooth}\}$.

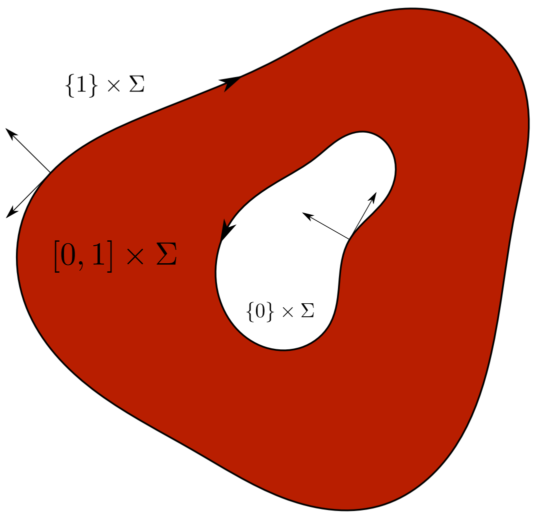
Proof. Due to the above considerations, the claim is equivalent to:
If $\hat{\gamma}([\omega]) = 0$ for all $\gamma : \mathbb{S}^1 \to M$ then $[\omega] = 0$.
So let $[\omega] \in H^1(M)$. Suppose $\int_\gamma \omega = 0$. Then ω is exact, i.e. $[\omega] = 0$. □

Definition 43. $f, \tilde{f} : \Sigma \to M$ *are called* homotopic*, if there is a smooth map* $F : [0,1] \times \Sigma \to M$ *such that*

$$
\begin{cases} F(0,p) = f(p) \\ F(1,p) = \tilde{f}(p) \end{cases} \tag{149}
$$

Theorem 41. *Let* $f, \tilde{f} : \Sigma \to M$ *are homotopic then* $f, \tilde{f}$ *are homologous.*

Proof. Furnish the disjoint boundary $\partial([0,1] \times \Sigma) = \{0\} \times \Sigma \coprod \{1\} \times \Sigma$ with the induced orientation of $[0,1] \times \Sigma$.



Choose $\omega \in \Omega^k(M)$ with $d\omega = 0$. Then using the usual Stokes argument leads to

$$0 = \int_{[0,1]\times\Sigma} F^* d\omega = \int_{[0,1]\times\Sigma} d(F^*\omega) = \int_\Sigma \tilde{f}^*\omega - \int_\Sigma f^*\omega$$

Hence $\hat{\tilde{f}} = \hat{f}$. □

Definition 44. *M is called* simply connected *if every $\gamma : \mathbb{S}^1 \to M$ is homotopic to a constant map.*

Theorem 42. *If M is simply connected then $H^1(M) = \{0\}$.*

Proof. For a constant loop $\eta : \mathbb{S}^1 \to M$ we have $\hat{\eta} = 0$. Due to the last theorem $\hat{\gamma} = 0$ for all $\gamma : \mathbb{S}^1 \to M$. □

Example 13.

- *$\mathbb{D}^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ has $H^1(\mathbb{D}^n) = 0$ because it is simply connected.*
- *By Sarde's lemma $\mathbb{S}^n$, $n \geq 2$ is simply connected and therefore $H^1(\mathbb{S}^n) = 0$.*

Classification of 3-dimensional manifolds up to diffeomorphisms is difficult.

Example 14. ***Poincaré conjecture*** *Every connected and simply connected 3-dimensional manifold without boundary is diffeomorphis to $\mathbb{S}^3$.*

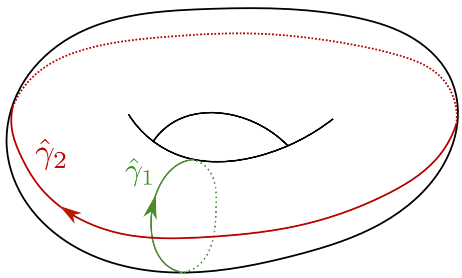
Surfaces

- Only $\mathbb{S}^2$ is simply connected without boundary.
- Only $\mathbb{D}^2$ is simply connected with $\partial M \neq \emptyset$.

- The *torus*

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{R}^2 / \mathbb{Z}^2 \tag{150}$$

is not simply connected.

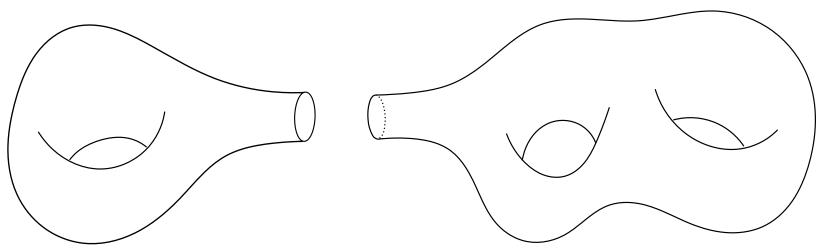


$\hat{\gamma}_1$, $\hat{\gamma}_1$ form a homology basis, so $H^1(\mathbb{T}^2) = 2$.
Constant 1-forms $\omega = adx + bdy$ are pullbacks of closed forms on $\mathbb{T}^2$ under the projection

$$\Pi : \mathbb{R}^2 \to \mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{T}^2$$

Hence $H^1(\mathbb{T}^2) = \{adx + bdy \mid a, b \in \mathbb{R}\}$.

Let $M, \tilde{M}$ be connected n-dimensional oriented manifolds.



Cut out a hole from M diffeomorphic to $\mathbb{D}^2$. Similar cut a hole in $\tilde{M}$. Glue hole boundaries to form another n-dimensional manifold $M\#\tilde{M}$ (connected sum of $M, \tilde{M}$).

Remark 4. *The diffeomorphism type of $M\#\tilde{M}$ is independent of how we cut holes and how we glue.*

Theorem 43.

$$H^k(M\#\tilde{M}) = H^k(M) \bigoplus H^k(\tilde{M}) \tag{151}$$

Theorem 44. *Every connected oriented compact 2-manifold without boundary is diffeomorphic to one of these $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{T}^2\#\cdots\#\mathbb{T}^2$.*

So $\dim H^1(M) = 2g$ for some $g \in \mathbb{N}$ and g is called the *genus* of M.

5.1 Vorticity from the Riemannian View Point

In this section we investigate vorticity expressed in the language of cohomology. For this purpose we recall the essential facts from section 4.3.
Let $\hat{M}, M$ be compact n-dimensional oriented manifolds with boundary. $\hat{M}$ comes with $\widehat{\det} \in \Omega(\hat{M})$ and M comes with a Riemannian metric $\langle \cdot, \cdot \rangle$. Moreover let $g_t : \hat{M} \to M$ be a diffeomorphism with $g_t^* \det = \widehat{\det}$ and $\dot{g}_t = v_t \circ g_t$ for some time dependent divergence free vector field $v_t \in \Gamma(TM)$, i.e. $\mathrm{div} v_t = 0 \Leftrightarrow \mathcal{L}_{v_t} \det = 0$.
Then the equation of motion in terms of $\eta = v^\flat$ reads

$$\dot{\eta}_t + \mathcal{L}_{v_t} \eta_t = -dp_t$$

for some family of functions $t \mapsto p_t \in \mathcal{C}^\infty(M)$. Furthermore we derived

$$\dot{\hat{\eta}}_t = -d\hat{p}_t$$

where $\hat{\eta} = g_t^* \eta_t$ and $\hat{p}_t = p_t \circ g_t$. This means "$\hat{\eta}$ modulo exact 1-forms is fixed in time" which implies

$$\dot{\hat{\omega}} = d\dot{\hat{\eta}} = 0$$

is fixed in time. For $\omega_t = g_t^{-1*} \hat{\omega}$ one can say it is

- "just advected"
- "flows with the fluid"
- "frozen in the fluis"

For each time t the euqivalence class

$$\eta + \{\text{exaxt} \;\; 1-\text{forms}\} = \Omega^1(M) \big/ \mathrm{Im}\, d_0 \tag{152}$$

determines v completely.

Theorem 45. *Given $u \in \Gamma(TM)$ then there is $q \in C^\infty(M)$ (unique up to an additive constant) such that*

$$u + \mathrm{grad}\, q \in \mathrm{sdiff}(M) \tag{153}$$

Proof.

$$u + \mathrm{grad}\, q \in \mathrm{sdiff}(M) \Leftrightarrow \begin{cases} \mathrm{div} u + \Delta q = 0 \\ \langle u + \mathrm{grad}\, q, N \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} \Delta q = -\mathrm{div} u \\ \langle \mathrm{grad}\, q, N \rangle = -\langle u, N \rangle \end{cases}$$

The Neumann problem for q is solvable because $\int_M -\mathrm{div}\; u = \int_{\partial M} -\langle u, N \rangle$ □

Taking the vorticity

$$\eta + \mathrm{Im} d_0 \mapsto \omega = d\eta$$

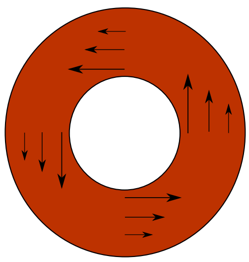
does not capture all information about $\eta + \mathrm{Im} d_0$.
$d\eta = d\tilde{\eta}$ only means $d(\eta - \tilde{\eta}) = 0$. In general it does not imlpy that

$$\begin{aligned} &\eta + \mathrm{Im} d_0 = \tilde{\eta} + \mathrm{Im} d_0 \\ &\Leftrightarrow \tilde{\eta} - \eta \in \mathrm{Im} d_0 \end{aligned}$$

So we want to understand how $v, \tilde{v} \in \mathrm{sdiff}(M)$ can be different even if $\omega = \tilde{\omega}$.

Definition 45. *$v \in \mathrm{sdiff}(M)$ is called a* harmonic *vector field if $dv^\flat = 0$.*

Example 15.



$$M := \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$$

$$v(x, y); = \tfrac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$$

Theorem 46. *For each cohomology class $[\alpha] \in H^1(M)$ there is a unique harmonic vector field $v \in \mathrm{sdiff}(M)$ such that $v^\flat = [\alpha]$.*

Proof. Given $\alpha \in \Omega^1(M)$ with $d\alpha = 0$. Then there is $q \in C^\infty(M)$ (unique up to an additive constant) such that $v = (\alpha + dq)^\sharp \in \mathrm{sdiff}(M)$. This v is harmonic since $dv^\flat = 0$. □

Definition 46. *$v \in \mathrm{sdiff}(M)$ is called a* potetial flow *if there is $q \in C^\infty(M)$ such that $v = \mathrm{grad}\, q$.*

Theorem 47. *There is no potential flow:*

$$v \text{ potential flow} \quad \Rightarrow \quad v = 0$$

Proof.

$$v = \mathrm{grad}\, q \quad \Rightarrow \quad q \text{ solves } \begin{cases} \mathrm{div}\,\mathrm{grad}\, q = 0 \\ \langle \mathrm{grad}\, q, N \rangle = 0 \end{cases}$$

Then $q \equiv 0$. The solution is unique up to a constant. Hence $v \equiv 0$. □

Remark 5. *Locally every flow with no vorticity ($dv^\flat = 0$) is a potential flow, if you forget about tangancy to the boundary. (If the boundary is at infinity then every closed form is exact)*

The question is: What are the conditions for $\omega \in \Omega^2(M)$, $d\omega = 0$ such that there exists $v \in \operatorname{sdiff}(M)$ with $\omega = dv^\flat$?

- v exists $\Leftrightarrow$ ω is exact.

But, if v exists, how many different v's are there?

- Difference of any two v's with the same vorticity ω, is a harmonic vector field. i.e.

$$H^1(M) \leftrightarrow \{\text{harmonic vector field}\}$$

Theorem 48. *Let $\Sigma_1, \ldots, \Sigma_k$ be the connected components of the boundary $\partial M = \Sigma_1 \amalg \cdots \amalg \Sigma_k$. Moreover define*

$$\begin{aligned} \alpha : \mathbb{R}^k &\to \mathbb{R} \\ (x_1, \ldots, x_k) &\mapsto x_1 + \cdots + x_k \end{aligned} \tag{154}$$

Then the map

$$\begin{aligned} \sigma : H^{k-1}(M) &\to \operatorname{Ker} \alpha \\ [\omega] &\mapsto \begin{pmatrix} \int_{\Sigma_1} \omega \\ \vdots \\ \int_{\Sigma_k} \omega \end{pmatrix} \end{aligned} \tag{155}$$

is well defined and surjetiv.

Proof.

- well defined: Since the connected components Σ_i of ∂M themself have no boundary, adding an exact $\eta \in \Omega^{k-1}(M)$ to the representive ω does not change the integral i.e.

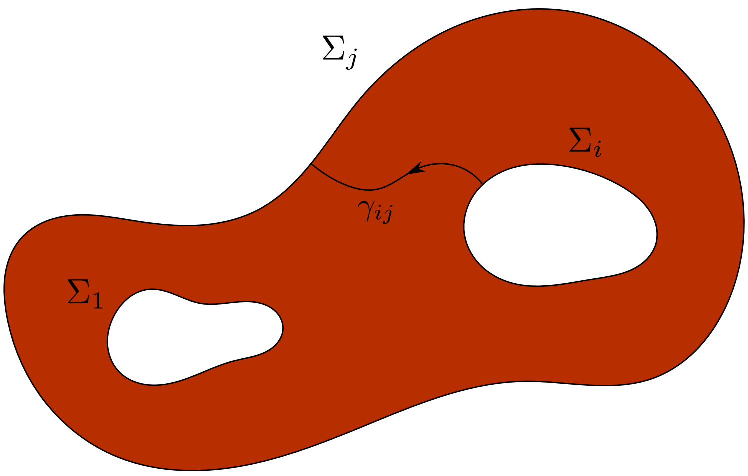
$$\int_{\Sigma_i} \omega = \int_{\Sigma_i} (\omega + \eta)$$

- surjective: For each $i, j \in \{1, \ldots, k\}$ choose a regular embedded curve

  $$\gamma_{ij} : [0,1] \to M$$

  such that

  $$\begin{aligned} \gamma_{ij}(0) \in &\Sigma_i & \gamma'_{ij}(0) \perp T_{\gamma_{ij}(0)}\Sigma_i \\ \gamma_{ij}(1) \in &\Sigma_j & \gamma'_{ij}(1) \perp T_{\gamma_{ij}(0)}\Sigma_j \end{aligned}$$



Construct a divergence free vector field X_{ij} supported in a neighbouhood of γ_{ij}, such that

$$\int_{\Sigma_l} \langle X_{ij}, N \rangle = \begin{cases} -1 & l = i \\ 1 & l = j \\ 0 & \text{otherwise} \end{cases}$$

For $\omega_{ij} := \iota_{X_{ij}} \det$ one obtains

$$\int_{\Sigma_l} \omega_{ij} = \begin{cases} -1 & l = i \\ 1 & l = j \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\rho([\omega_{ij}]) = e_{ij} \qquad e_{ij} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1_i \\ 0 \\ \vdots \\ 0 \\ 1_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now use $\operatorname{span}\{e_{ij} \mid i \neq j\} = \operatorname{Ker} \alpha$

□

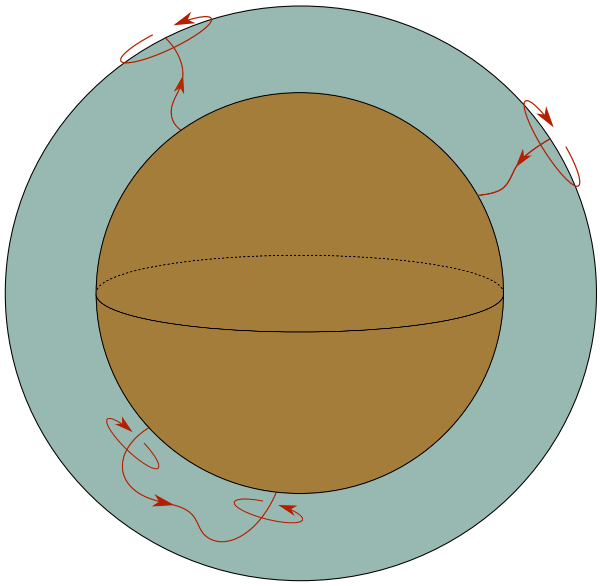
Theorem 49. *If M is a domain in $\mathbb{R}^n$ then ρ is injective.*

Corollary 3. *Let M be a compact domain in $\mathbb{R}^n$ with smooth boundary. Then $\omega \in \Omega^{k-1}(M)$ is exact if and only if*

$$\int_{\Sigma_1} \omega = \cdots = \int_{\Sigma_k} \omega = 0$$

Example 16. $\dim M = 3$

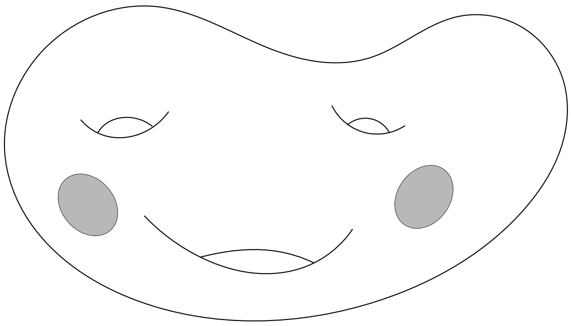
- *Think of the earth with atmosphere.*



Corollary 3. tells us, there can not be a single tornado that tuches both, the earth and the heaven.

- *A spoon in a coffee cup. Vortexes arise at surfaces only.*

Example 17. $\dim M = 2$



Let M be compact oriented manifold with boundary $\Sigma_1, \dots, \Sigma_k$. Glue a disc to each Σ_i. This gives us $\hat{M} \simeq M_g$. So every compact oriented surfaces with boundary arised from a surface $\hat{M}$ without boundary by deleting k disc-shaped holes. $M \simeq M_{g,k}$

Theorem 50. $\dim H^1(M_{g,k}) = 2g + k - 1$
In particular, if M is a domain in $\mathbb{R}^2$ then $\dim H^1(M) = k - 1$.

6 Fluid Dynamics of Surfaces

Definition 47.

- *A bilinear form $\sigma : V \times W \to \mathbb{R}$ is called* non-degenerate *if*

$$\begin{cases} \sigma(X, Y) = 0 \quad \forall Y \in W \quad \Rightarrow \quad X = 0 \\ \sigma(X, Y) = 0 \quad \forall X \in V \quad \Rightarrow \quad Y = 0 \end{cases} \tag{156}$$

- *Let M be a manifold. Then $\sigma \in \Omega^2(M)$ is called* symplectic form *if σ_p is non-degenerate for each $p \in M$ and $d\sigma = 0$.*

Remark 6. *If σ is a non-degenerated 2-form then $\dim M = 2k$ and $\underbrace{\sigma \wedge \cdots \wedge \sigma}_{k} \neq 0$ is a volume form.*

Definition 48. *Let σ be a symplectic form.*

- *A vector field $X \in \mathrm{sp}(M) := \{Y \in \mathrm{diff}_0(M) \mid \mathcal{L}_X \sigma = 0\}$ is called* symplectic.
- *A map $g : M \to M$ is called* symplectic *if*

$$g^* \sigma = \sigma$$

So: X is a symplectic vector field on a compact oriented manifold M if and only if $\exp(tX)$ is symplectic for all $t \in \mathbb{R}$.

Theorem 51.

$$X \in \Gamma(TM) \text{ symplectic} \quad \Leftrightarrow \quad d(\iota_X \sigma = 0) \tag{157}$$

Proof.

$$X \text{ symplectic } \Leftrightarrow 0 = \mathcal{L}_X\sigma \overset{Cartan}{=} d(\iota_X\sigma) + \iota_X \underbrace{d\sigma}_{=0}$$

□

Definition 49. *A vector field $X \in \mathrm{sp}(M)$ is called* Hamiltonian *if $\iota_X\sigma = -dH$ for some $H \in C^\infty(M)$.*

Definition 50. *Let M be a symplectic manifold, then the* symplectic gradient

$$\mathrm{sgrad} f \in \Gamma(TM) \tag{158}$$

is defined by

$$\iota_{\mathrm{sgrad} f}\sigma = -df \tag{159}$$

Intuition 2. *Analogoulsly one can define the ordinary gradient as*

$$Y = \mathrm{grad}\, f$$

the unique vector field such that

$$\langle X, Y \rangle = df(X) \quad \forall X \in \Gamma(TM)$$

So now we define

$$Y = \mathrm{sgrad} f$$

the unique vector field such that

$$\sigma(X, Y) = df(X) \quad \forall X \in \Gamma(TM)$$

Remark 7. *If $H^1(M) = 0$ (every closed 1-form is exact, like B_n or S^n) then every symplectic vector field is Hamiltonian.*

Example 18. *Let $M = \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ be a torus and $\sigma = dx \wedge dy$, $X = \begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in \mathbb{R}$ its symplectic form and vector field respectively.*

$$\begin{aligned} H \in C^\infty(\mathbb{T}^2) \qquad H : \mathbb{R}^2 \to \mathbb{R}^2 \\ H(x + 2\pi, y) = H(x, y) \\ H(x, y + 2\pi) = H(x, y) \end{aligned}$$

then

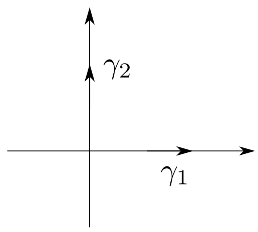
$$\underbrace{-\left(\frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy\right)\begin{pmatrix} v \\ w \end{pmatrix}}_{-\frac{\partial H}{\partial x}v - \frac{\partial H}{\partial y}w} = \sigma\left(\begin{pmatrix} a \\ b \end{pmatrix}; \begin{pmatrix} v \\ w \end{pmatrix}\right) \overset{\text{just\_det}}{=} aw - bv$$

$$\Rightarrow \quad \frac{\partial H}{\partial y} = -a \qquad \frac{\partial H}{\partial x} = b$$

Thus

$$\begin{pmatrix} a \\ b \end{pmatrix} = -J \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} = -J \text{sgrad} H$$

and



$$0 = \int_{\gamma_2} \frac{\partial H}{\partial y} = -2\pi a \quad 0 = \int_{\gamma_2} \frac{\partial H}{\partial x} = 2\pi b \quad \rightsquigarrow \quad X = 0$$

So X is symplectic but not Hamiltonian.

Theorem 52. *Let $X, Y \in \text{sp}(M)$ be symplectic vector fields. Then $[X, Y] \in \text{sp}(M)$ and $[X, Y] = \text{sgrad}\sigma(X, Y)$. "The Lie bracket of two symplectic vector fields is always Hamiltonian".*

Proof.

1.

$$\begin{aligned} 0 = d\sigma(X, Y, Z) =& X\sigma(Y, Z) + Y\sigma(Z, X) + Z\sigma(X, Y) \\ & - \sigma([X, Y], Z) - \sigma([Y, Z], X) - \sigma([Z, X], Y) \end{aligned}$$

2.

$$\begin{aligned} 0 = (\mathcal{L}_X\sigma)(Y, Z) &= \mathcal{L}_X(\sigma(Y, Z)) - \sigma(\mathcal{L}_X Y, Z) - \sigma(Y, \mathcal{L}_X Z) \\ &= X\sigma(X, Y) - \sigma([X, Y], Z) - \sigma(Y, [X, Z]) \end{aligned}$$

3.

$$0 = (\mathcal{L}_Y\sigma)(Z, X) = Y\sigma(Z, X) - \sigma([Y, Z], X) - \sigma(Z, [Y, X])$$

$$1. - 2. - 3. = 0 = Z\sigma(X, Y) + \sigma(Z, [Y, X]) = (d\sigma(X, Y))(Z) + \underbrace{\sigma([X, Y], Z)}_{\iota_{[X,Y]}\sigma(Z)}$$

$$\rightsquigarrow \quad \iota_{[X,Y]}\sigma = -d\sigma(X, Y)) \quad \rightsquigarrow \quad \text{sgrad}\sigma(X, Y) = [X, Y]$$

□

Theorem 53. *Let $H, \tilde{H} \in C^\infty$ be smooth functions and $X := \mathrm{sgrad} H$, $\tilde{X} := \mathrm{sgrad} \tilde{H}$. then the following are equivalent*

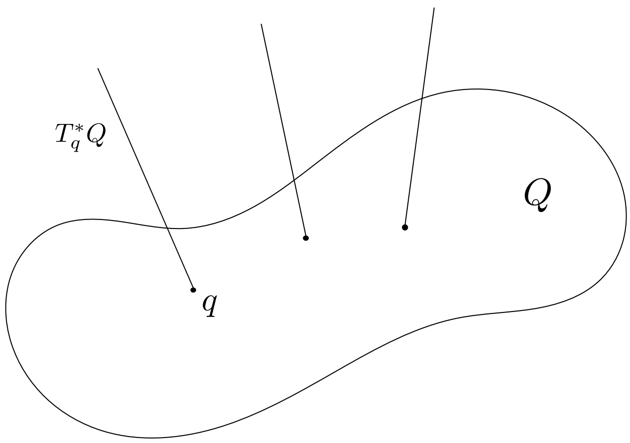
1. $\mathcal{L}_X \tilde{H} = 0$
2. $\mathcal{L}_{\tilde{X}} H = 0$
3. $\sigma(X, \tilde{X}) = 0$

Proof.

$$0 = \mathcal{L}_X \tilde{H} = d\tilde{H}(\mathrm{sgrad}\, H) = -\sigma(\mathrm{sgrad}\, H, \mathrm{sgrad}\, \tilde{H}) = \cdots = -\mathcal{L}_{\tilde{X}} H$$

□

In addition we have $\sigma(X, \tilde{X}) = 0 \rightsquigarrow [X\tilde{X}] = \mathrm{sgrad}\, \sigma(X, \tilde{X}) = 0$.
The main example in physics of a symplectic manifold is $M = T^*Q$, where Q is an arbitrary manifold.



Let $\pi : T^*Q \to Q$ be the projection map and $\alpha \in \Omega^1(T^*Q)$ given by

$$\alpha_\omega(X) := \omega(\underbrace{d\pi(X)}_{\in T_{\pi(\omega)}Q}) \tag{160}$$

where $X \in T_\omega M$ and $\omega \in T_q^*Q$. Or in another notation

$$T_\omega M \ni X \overset{\alpha}{\mapsto} \omega(d\pi(X)) \in \mathbb{R} \tag{161}$$

Locally we have coordinates

$$\begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{pmatrix} : U \to \mathbb{R}^n \quad \text{on} \quad Q$$

Define $q_1 := \hat{q}_1 \circ \pi, \ldots, q_n := \hat{q}_n \circ \pi \in C^\infty(\pi^{-1}(U))$. Then $\omega \in T_q^*Q$

$$\omega = p_1 d\hat{q}_1|_{T_qQ} + \cdots + p_n d\hat{q}_n|_{T_qQ} \quad \rightsquigarrow \quad p_1, \ldots, p_n : \pi^{-1}(U) \to \mathbb{R}$$

So $p_1, \ldots, p_n, q_1, \ldots q_n$ are really coordinates on $\pi^{-1}(Q)$

$$\alpha = p_1 dq_1, \ldots, p_n dq_n$$

Theorem 54. *$\sigma := d\alpha$ is a cannonical symplectic form on T^*Q.*

Proof. In the above coordinates $\sigma = dp_1 \wedge dq_1, \ldots, dp_n \wedge dq_n$ is non-degenerate. □

Main example for H is $\langle \cdot, \cdot \rangle$ Riemannian metric on Q (kinetic energy) and $V \in C^\infty(Q)$ (potential energy).

Identify T^*Q with TQ via $\langle \; , \; \rangle$.

Theorem 55. *$\gamma : [a, b] \to M = TQ$ is an integral curve of $-\operatorname{sgrad} H$ if and only if $\hat{\gamma} = \pi \circ \gamma : [a, b] \to Q$ satisfies*

$$\hat{\gamma}''(t) = -\operatorname{grad}_{\hat{\gamma}(t)} V \tag{162}$$

Proof. Let $X \in TQ$ be an element of the tangent bundle. Near q choose a vector field Y on Q with

$$Y_q = X \qquad \nabla_Z Y = 0 \qquad \forall Z \in T_q Q$$

and define

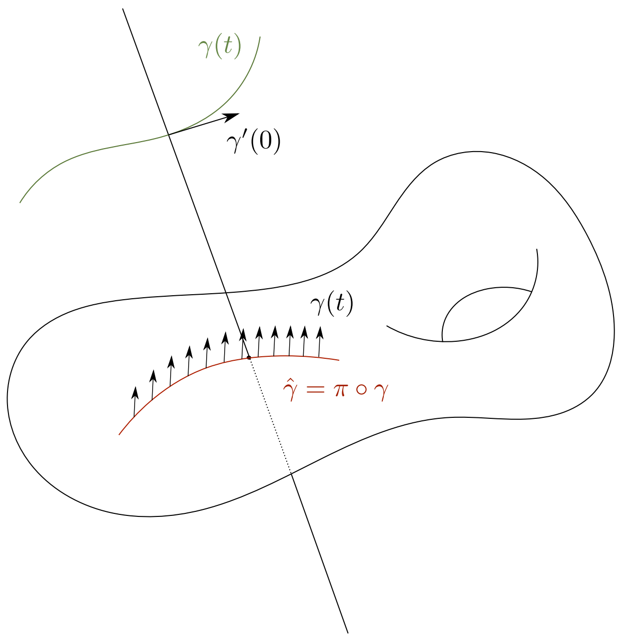
$$H_X := \{dY(Z) \mid Z \in T_q M\}$$

Then we split the tangent space into

$$T_X M = \underbrace{T_X(T_q Q)}_{\text{vertical space}} \bigoplus \underbrace{H_X}_{\text{horizontal space}}$$

Let $\gamma : [-\varepsilon, \varepsilon] \to M$ be a curve in M with $\gamma(0) = X$. Then

$$\underbrace{\gamma'(0)}_{\in T_X M} = \underbrace{\frac{D\gamma}{dt}(0)}_{\in T_X(T_q Q)} + \underbrace{\gamma_H}_{\in H_X}$$



We can identify $T_X(T_qM) \simeq T_qQ$ via the vector space isomorphism

$$d_X\pi|_{H_X} : H_X \to T_qM$$

Define a Riemannian metric on M by demanding that $V_X \perp H_X$. $V_X \simeq T_qM$ already has a metric, H_X gets a metric by asking that $T_X(T_qM) \simeq T_qQ$ is an isometrie.
In other words: Both V and H as rank n vector bundles over M can be identified with $E := \pi^*(TQ)$ i.e. $TM = E \bigoplus E$. So they come with the pullback of the Levi-Civita connection.
Now let ω be a tautological 1-form on Q, $\omega(Y) = Y \rightsquigarrow d^{\nabla}\omega = 0$ since ∇ is torsion free.
Rename π to $\mathfrak{q} \rightsquigarrow$ "$d\mathfrak{q}$" $\in \Omega^1(E)$, $d\mathfrak{q}(Z) = \pi(Z)$ for $Z \in T_XM$ and $\mathfrak{q}(X) = q$ since $\omega(X) = X$. With this notation α becomes $\alpha = \langle\, p, d\mathfrak{q}\,\rangle$. Therefore $\sigma = \langle\, dp \wedge d\mathfrak{q}\,\rangle$ where

$$\langle\, \eta \wedge \nu\,\rangle(X,Y) = \langle\, \eta(X), \nu(Y)\,\rangle - \langle\, \eta(Y), \nu(X)\,\rangle \tag{163}$$

We would like to determine $\operatorname{sgrad}_X H = \begin{pmatrix} Y_V \\ Y_H \end{pmatrix}$ So

$$\begin{aligned}
\langle\, X, Z_V\,\rangle + \langle\, \operatorname{grad}_q V, Z_H\,\rangle &= d_X H \begin{pmatrix} Z_V \\ Z_H \end{pmatrix} \\
&= -(\iota_{\begin{pmatrix} Y_V \\ Y_H \end{pmatrix}} \sigma) \begin{pmatrix} Z_V \\ Z_H \end{pmatrix} = -\sigma\left(\begin{pmatrix} Y_V \\ Y_H \end{pmatrix}, \begin{pmatrix} Z_V \\ Z_H \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} Z_V \\ Z_H \end{pmatrix}, \begin{pmatrix} Y_V \\ Y_H \end{pmatrix}\right) = \langle\, Z_V, Y_H\,\rangle - \langle\, Z_H, Y_V\,\rangle
\end{aligned}$$

$$\rightsquigarrow \begin{cases} Y_H = X \\ Y_V = -\operatorname{grad}_q V \end{cases}$$

We now apply the result to $\hat{\gamma} := \pi \circ \gamma$ then

$$\hat{\gamma}' = Y \qquad \rightsquigarrow \qquad \hat{\gamma}'' = -\operatorname{grad}_{\hat{\gamma}(t)} V$$

□

If $\dim M = 2$ then the volume form $\det =: \sigma$ is a symplectic form and the velocity field $v \in \operatorname{sdiff}(M)$ is symplectic. Moreover if M is a toplogical $\mathbb{S}^2$ or $M \subset \mathbb{R}^2$ then V is automatically Hamiltonian.

$$X \in T\partial M \quad \rightsquigarrow \quad \omega(X) = \sigma(v, X) = 0 \quad \rightsquigarrow \quad \omega|_{T\partial M}$$

Therfore we have $\int_{\Sigma_k} \omega = 0$ for all boundary components. Hence $\omega = dH$ and $v = -\operatorname{sgrad} H$.
$H \in C^\infty(M)$ is a constant of motion.

$$\mathcal{L}_v H = v \cdot H = dH(v) = \sigma(v, v) = 0$$

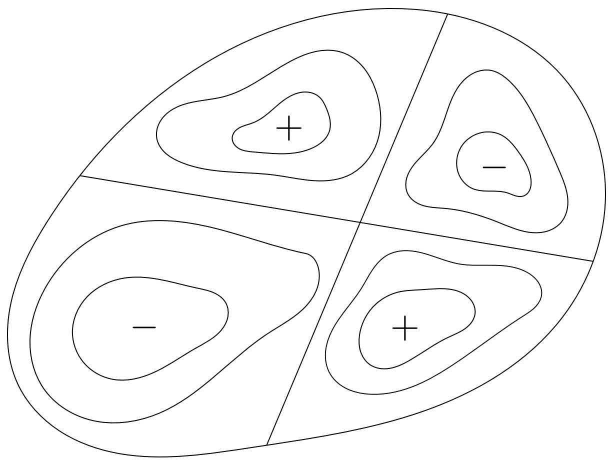
Symplectic Flows

- 2D fluidmechanics $\rightsquigarrow$ 3D stationary flows
- Q Riemannian, $V \in C^\infty(M)$
  $\rightsquigarrow \ddot{\gamma} = -\mathrm{grad} V$ Hamiltonian
  - compressible fluid
    $Q = \{g : \hat{M} \to M \mid g \text{ diffeo}\}$
    $V = \int_{\hat{M}} W\left(\frac{1}{\det g'}\right)$
  - incompressible fluid
    $Q = \{g : \hat{M} \to M \mid \det g' = 1\}$
    no V
- Schrödinger flow
  $\rightsquigarrow$ incompressible Schrödinger flow

The flow of a divergence free vector field v on a 2-dim surface is orderly. At least on $\mathbb{S}^2$ or planar domains

$$v = \mathrm{sgrad}\,\phi$$

where ϕ is the "steam function". If $\mathcal{L}_v\phi = 0$ then ϕ is a constant of motion and the flow lines are level lines of $\phi \rightsquigarrow$ almost all flow loines are closed.



A divergence free vector field v on a 3-dim manifold M can have a chaotic flow. Let

$M = \Sigma \times \mathbb{S}^1$ and $u_t \in \text{sdiff}(\Sigma)$ for all $t \in \mathbb{S}^1$. Then define

$$v_{(q,t)} := \begin{pmatrix} u_t(q) \\ 1 \end{pmatrix}$$

Let σ be the volume form on Σ then $\det = \sigma \wedge dt$ is the volume form on M.
Start with $\begin{pmatrix} q \\ 0 \end{pmatrix} \in M$, $q \in \Sigma$ then define

$$g_{2\pi} \begin{pmatrix} q \\ 0 \end{pmatrix} = \begin{pmatrix} F(q) \\ 2\pi(\equiv 0) \end{pmatrix}$$

$F : \Sigma \to \Sigma$ is called the *Poincaré map* of v. F is a symplectic map, since

$$(F^*\sigma)(X, Y) = \sigma(dF(X), dF(Y)) = \sigma(X, Y)$$

We are now concerned with some numericel methods.
Verlet method
The Verlet method works for equations of the form

$$\ddot{u} = f(u) \tag{164}$$

In order to approximate $f(nh)$ by u_n we use the mean value theorem twice

$$\begin{aligned} \ddot{u}(t) &\approx \frac{\dot{u}(t+\frac{h}{2}) - \dot{u}(t-\frac{h}{2})}{h} \approx \frac{\frac{u(t+h)-u(t)}{h} - \frac{u(t)-u(t-h)}{h}}{h} \\ &= \frac{u(t+h) - 2u(t) + u(t-h)}{h^2} \end{aligned}$$

This gives us the following recursion formula for the u_n:

$$\begin{aligned} f(u_n) &= \frac{u_{n-1} - 2u_n + u_{n+1}}{h^2} \\ \Leftrightarrow u_{n+1} &= 2u_n - u_{n-1} - h^2 f(u_n) \end{aligned}$$

If we consider that as a two dimensional system, we have

$$\begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} \overset{F}{\longmapsto} \begin{pmatrix} u_n \\ 2u_n - u_{n-1}h^2 f(u_n) \end{pmatrix}$$

In particular

$$F'(x, y) = \begin{pmatrix} 0 & 1 \\ -1 & 2 - h^2 f'(y) \end{pmatrix} \quad \rightsquigarrow \quad \det F' = 1$$

Thus F is volume preserving.

Example 19. *(Symplectic map $F : \mathbb{T}^2 \to \mathbb{T}^2$)*
Let $U : \mathbb{R} \to \mathbb{S}^1$ and $f : \mathbb{S}^1 \to \mathbb{R}$ satisfying $u'' = -f(u)$. For $f(u) = \sin u$ this is the pendulum equation.
Start with a discetisation $u_0, u_1, \ldots$ given by $u_n = u(n \cdot h)$ for some time step $h > 0$. Compare the following methods for some initial data.

- *Use a Runge-Kutta method and draw 1000 points of the form $\begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}$.*
- *For $p := \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ draw $p, F(p), F^2(p), \ldots, F^{10000}(p)$.*

6.1 Sationary Flows in 3D

For stationary flows the equation of motion become

$$\begin{cases} v \times \mathbf{w} = \operatorname{grad} \alpha \\ [v, \mathbf{w}] = 0 \end{cases}$$

We distinguish between the following cases

- α is non constant
  Recall that the Lie derivative of a function is just the directional derivative. Therefore

  $$\mathcal{L}_v \alpha = \langle \operatorname{grad} \alpha, v \rangle = \langle v \times \mathbf{w} \rangle = 0$$

  Thus the Bernoulli pressure α is a constant of motion.

- α is constant
  Then $v \times \mathbf{w} = 0$, hence $\mathbf{w} = \lambda v$ locally and $0 = [v, \lambda v] = (\mathcal{L}_v \lambda) v$.
  - λ is not constant $\rightsquigarrow$ everything is orderly
  - the only possily chaotic case is:

    $$\text{curl} v = \lambda v \qquad \text{with} \qquad \lambda \in \mathbb{R}$$

    Then the vector field v is called *Beltrami field.*

  Example 20. *On* $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ *the Beltrami field with* curl$v = v$, div$v = 0$

  $$v(x, y, z) := \begin{pmatrix} A \sin z + C \cos y \\ B \sin x + A \cos z \\ C \sin y + B \cos x \end{pmatrix} \tag{165}$$

  is called the ABC-flow *(Arnold-Beltrami-Childress flow).*

7 Numerical Methods

1.

$$\dot{\eta} + \mathcal{L}_v \eta = -d\tilde{p}$$

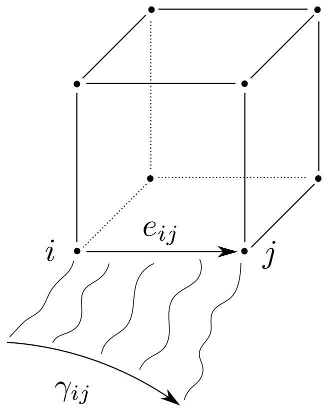
Devide a box M into cubical *voxels* of side length h. Model v as the "flux 2-form" $\iota_v \det$. For each oriented cube face φ assign a number $v_\varphi \cdot h$ supposed to model

$$\int_\varphi \iota_v \det = \int_\varphi \langle v, N \rangle$$

$\varphi_{i,j}$, $(i, j) \in \mathbb{Z}^3$ differ by 1 in just 1 coordinate (neighbouring voxels) $v_{ij} = -v_{ji}$. The divergence free condition becomes $\sum_j v_{ij} = 0$. Draw an edge e_{ij} from the center of voxel i to the center of each neighbouring voxel j and assign a number $v_{ij} \cdot h = \eta_{ij}$

supposed to model $\int_{e_{ij}} \eta$.
Velocity is modeled as a *staggered vector field*:
component v_x, v_y, v_z given on face centers facing in x, y, z-direction respectively and $v_\varphi = 0$ on boundary faces.
The velocity is linearly interpolated to all points of M component wise.
• Time evolution of η:
Given η_t at time t. Use "time splitting" to compute $\eta_{t+\delta}$

a) First solve $\dot{\eta} + \mathcal{L}_v \eta = 0$ over the time interval $[t, t\delta]$ with v fixed to v_t.



Use the Kelvin circulation theorem. Flow edge e_{ij} backward in time over the time step leads to the curve γ_{ij}. $\rightsquigarrow \int_{e_{ij}} \eta_{t+\delta} = \int_{\gamma_{ij}} \eta_t$.

b) Add $-dp$ to η to reestablish div$v = 0$.

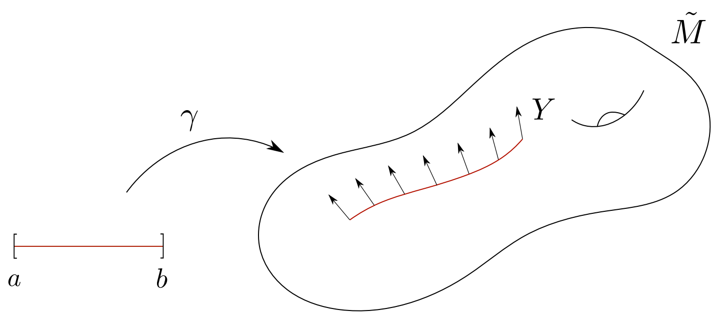
2.

$$\dot{v} + \nabla_v v = -\operatorname{grad} p$$

Model v as $v_i \in \mathbb{R}^3$ at each voxel center. Define the flux through face φ_{ij} as $h^2 \frac{\langle v_i, N \rangle + \langle v_j, N \rangle}{2}$. In the advection step advect v as an $\mathbb{R}^3$-valued function.
Given a voxelcenter i, find that point $p \in M$ that flows to i within time δ, based on the current v_t. Then assign to i the value $v_t(p)$.

7.1 Schrödinger Flow

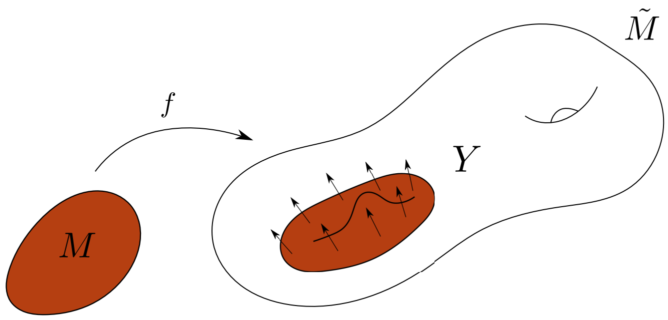
In order to prepare ourselves for the Schrödinger flow we derive another version of variational formula.
Let $\tilde{M}$ be a Riemannian manifold and let $Y \in \Gamma(\gamma^* T\tilde{M})$ with $\dot{\gamma} = Y$ be a variational vector field.



Then the first variational formula of the Dirichlet energy $E(\gamma) = \frac{1}{2}\int_a^b |\gamma'|^2$ (compare Theorem 8) is given by

$$\dot{E} = \langle\, Y, \gamma' \,\rangle \,|_a^b - \int_a^b \langle\, Y, \gamma' \,\rangle$$

We now consider $f : M \to \tilde{M}$ where M is a compact Riemannian manifold with boundary.



Let V, W vector spaces od finite dimension and $A : V \to W$ a linear map. Then we define

$$|A|^2 := \mathrm{tr}(A^* A) \tag{166}$$

If $v_1, \dots, v_n$ is a orthnonormal basis of V and $w_1, \dots, w_m$ is a orthnonormal basis of W then $Av_j = \sum_{i=1}^{M} a_{ij} w_j$ and therefore

$$|A|^2 = \sum_{j=1}^{n} \langle\, A^* A v_j, v_j \,\rangle = \sum_{j=1}^{n} \langle\, Av_j, Av_j \,\rangle = \sum_{j=1}^{n} |Av_j|^2 = \sum_{i,j} a_{ij}^2$$

On a Riemannian manifold M, locally there exists a orthnonormal vector field $X_2, \dots, X_n$

an we can write $v \in \Gamma(TM)$ as $v = \sum_{j=1}^n v_j X_j$. Hence

$$\begin{aligned}\mathrm{div} v &= \sum_j \langle \nabla_{X_j} v, X_j \rangle = \sum_{i,j} \langle \nabla_{X_j}(v_j X_i), X_j \rangle \\ &= \sum_{i,j} (X_j v_j) \underbrace{\langle X_i, X_j \rangle}_{=\delta_{ij}} + \sum_{i,j} v_i \underbrace{\langle \nabla_{X_j} X_i, X_i \rangle}_{=\mathrm{div} X_i} \\ &= \sum_j X_j v_j + \sum_i v_i \,\mathrm{div} X_i\end{aligned}$$

If we apply this to the gradient of a function $f \in C^\infty(M)$ then

$$\Delta f = \mathrm{div}\,\mathrm{grad}\, f = \sum_i X_i X_i f + \sum_i (X_i f) \mathrm{div} X_i$$

Locally at a fixed $p \in M$ one can assume that $(\nabla X_j)_p = 0$ and hence $\mathrm{div} X_i = 0$. So

$$\Delta f = \sum_i X_i X_i f$$

More genarally let E qa vector bundle over M with Levi Civita connection ∇. Then we define for $\Psi \in \Gamma(E)$

$$\Delta \Psi = \sum_i \nabla_{X_i} \nabla_{X_i} \Psi + \sum_i (\mathrm{div} X_i) \nabla_{X_i} \Psi \tag{167}$$

This is independent of the the choice of $X_i, \dots, X_n$.
If $f : M \to \tilde{M}$ is defined as above and $\tilde{\nabla}$ is the Levi Civita connection on $\tilde{M}$. Then

$$\begin{cases} \Delta f \in \Gamma(f^* T\tilde{M}) \\ \Delta f = \sum_i \hat{\nabla}_{X_i} df(X_i) + \sum_i (\mathrm{div} X_i) df(X_i) \end{cases} \tag{168}$$

where $\hat{\nabla} = f^* \tilde{\nabla}$. Δf is called the *tension field* of f.

We now consider a variation $\dot{f} = Y$. Then

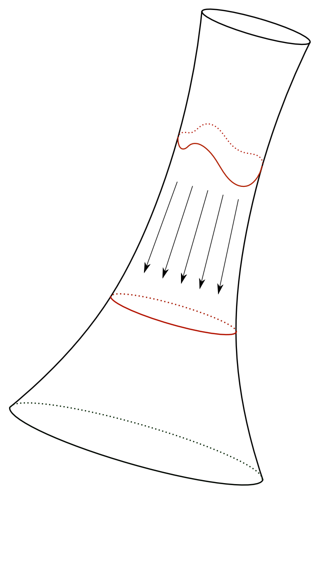
$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E(f_t) &= \frac{d}{dt}\Big|_{t=0} \frac{1}{2} \int_M \sum_i \langle\, df(X_i), df(X_i)\,\rangle \\
&= \sum_i \int_M \langle\, \frac{\partial}{\partial t} X_i f, df(X_i)\,\rangle \\
&= \sum_i \int_M \langle\, \hat{\nabla}_{X_i} Y, df(X_i)\,\rangle \\
&= \sum_i \int_M \Big(X_i \langle\, Y, df(X_i)\,\rangle - \langle\, Y, \hat{\nabla}_{X_i} df(X_i)\,\rangle \Big) \\
&= \sum_i \int_M X_i \langle\, df^*(Y), X_i\,\rangle - \int_M \langle\, Y, \Delta f\,\rangle \\
&= \int_M \operatorname{div}(df^*(Y)) - \int_M \langle\, Y, \Delta f\,\rangle \\
&\overset{\text{div Thm}}{=} \int_{\partial M} \langle (df^*(Y), N\,\rangle - \int_M \langle\, Y, \Delta f\,\rangle \\
&= \int_{\partial M} \langle\, Y, df^*(N)\,\rangle - \int_M \langle\, Y, \Delta f\,\rangle
\end{aligned}
$$

Definition 51. *$f : M \to \tilde{M}$ is called a* harmonic map *if f is a critical point of the Dirichlet energy with respect to a variation supported in $\mathring{M}$, i.e. $\Delta f = 0$.*

Example 21. *Let M be a manifold of dimension n. Then an immersion $f : M \to \mathbb{R}^{n+1}$ has constan mean curvature if and only if the Gauss-map $N : M \to \mathbb{S}^n$ is harmonic.*

Definition 52. *A family of smooth maps $f_t : M \to \tilde{M}$, $t \in [a, b]$ solves the* heat equation *if $t \mapsto f_t$ is an integral curve of $-\operatorname{grad} E$.*

Example 22. *In case of $M = \mathbb{S}^1$ use the heat equation to prove: If $\tilde{M}$ is compact with sectional curvature $K \leq 0$ then each homotopy class of loops in $\tilde{M}$ contains a a unique closed geodesic.*



A complex vector space V cqan be viewed as a real vector space V together with $J \in \mathrm{End}(V)$ with $J^2 = -id$.

Definition 53. *An* almost complex *manifold is a manifold M together with $J \in \mathrm{End}(V)$ with $J^2 = -id$.*

Definition 54. *An almost complex manifold which is also Riemannian is called* almost Kähler *id $J^* = -J$*

Remark 8. *In an almost Kähler manifold holds*

$$\langle JX, JY \rangle = \langle X, J^*JY \rangle = \langle X, Y \rangle$$

Definition 55. *An almost Kähler manifold is called* Kähler *if $\nabla J = 0$, i.e. $\nabla_X(JY) = J\nabla_X Y$ for all $X, Y \in \Gamma(TM)$.*

Theorem 56. *M Kähler $\Rightarrow$ M complex.*

Example 23. *Every oriented 2-manifold is Káhler.*

On every Kähler manifold there is $\sigma \in \Omega^2(M)$ defined by

$$\sigma(X, Y) = \langle JX, Y \rangle \tag{169}$$

This σ is symplectic, since it is obviouly skew symmetric and non-degenerate. Moreover for parallel vector fields $(\nabla X)_p = (\nabla Y)_p = (\nabla Z)_p = 0$ we have

$$\begin{aligned}(d\sigma(X, Y, Z))_p &= (X\sigma(Y, Z) + Y\sigma(Z, X) + Z\sigma(x, y))_p \\ &= (X \langle JY, Z \rangle + Y \langle JZ, X \rangle + Z \langle JX, Y \rangle)_p \\ &= (\langle \nabla_X(JY), Z \rangle + \langle JY, \underbrace{\nabla_X Z}_{=0} \rangle + \langle \nabla_Y(JZ), X \rangle + \langle JZ, \underbrace{\nabla_X Z}_{=0} \rangle + \cdots)_p \\ &= (\langle J\underbrace{\nabla_X Y}_{=0}, Z \rangle + (\langle J\underbrace{\nabla_Y Z}_{=0}, X \rangle + (\langle J\underbrace{\nabla_Z X}_{=0}, Y \rangle)_p = 0\end{aligned}$$

if $\tilde{M}$ is Kähler then also $\mathcal{M} = C^\infty(M, \tilde{M})$ is Kähler. For $Y \in T_f\mathcal{M} = \Gamma(f^*T\tilde{M})$ JY is already defined and also for $Y, Z \in T_f\mathcal{M}$

$$\tilde{\sigma}(Y, Z) = \int_M \sigma(Y, Z) \tag{170}$$

With some work $\tilde{\sigma}$ is closed in a appropriate sense.
The symplectic gradient flow of the Dirichlet energy E

$$\begin{cases} \dot{f} = J\Delta f \\ i\dot{\psi} = -\Delta\psi \end{cases} \tag{171}$$

gives us the Schrödinger equation.

Example 24.

- *$\tilde{M} = \mathbb{C}^m \rightsquigarrow$ ordinary Schrödinger equation.*
- *$\tilde{M} = \mathbb{S}^2$ Then for $Y \in T_S\mathbb{S}^2$ we have $-J = S \times Y$ and we obtain*

$$\dot{S} = S \times \Delta S \tag{172}$$

  the Landau-Lifschitz equation*.*

7.2 Incompressible Schrödinger Flow

Let M be a Riemannian manifold and let $\rho : M \to \mathbb{R}$ be a density. Moreover we define $\psi : M \to \mathbb{C}^n$ satisfying the Schrödinger equation $\dot{\psi} = \frac{i}{2}\Delta\psi$ and $\rho = |\psi|^2$. Via the Madelung transform one can define the velocity

$$\eta = \frac{\langle\, d\psi, i\psi \,\rangle}{|\psi|^2} \tag{173}$$

where $v = \eta^\sharp$ and $\langle\, \cdot, \cdot \,\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ is the ordinary scalar product.

Theorem 57.

$$\begin{aligned} &(\rho \det)^{\cdot} + \mathcal{L}_v(\rho \det) = 0 \\ \Leftrightarrow\ &\dot{\rho} \det + \langle\, \operatorname{grad} \rho, v \,\rangle \det + \rho(\operatorname{div} v) \det = 0 \\ \Leftrightarrow\ &\dot{\rho} + \operatorname{div}(\rho v) = 0 \end{aligned} \tag{174}$$

Proof.

$$\dot{\rho} = 2\,\langle\, \dot{\psi}, \psi \,\rangle = \langle\, i\Delta\psi, \psi \,\rangle = -\,\langle\, \Delta\psi, i\psi \,\rangle$$

$$\begin{aligned} \operatorname{div}(\rho v) &= \operatorname{div}(\langle\, d\psi, i\psi \,\rangle^\sharp) = \sum_j X_j \,\langle\, \underbrace{d\psi(X_j)}_{X_j\psi}, i\psi \,\rangle \\ &= \sum_j \left(\langle\, X_j X_j \psi, i\psi \,\rangle + \underbrace{\langle\, X_j\psi, iX_j\psi \,\rangle}_{=0} \right) \\ &= \langle\, \Delta\psi, i\psi \,\rangle \end{aligned}$$

$$\rightsquigarrow \quad \dot{\rho} + \operatorname{div}(\rho v) = 0$$

□

We want incompressible fluids. So we impose the constraint $|\psi| = 1$ ($\rightsquigarrow$ $\operatorname{div} v = 0$). Suppose at time $t = 0$ we have ψ with $|\psi| = 1$ and $0 = \operatorname{div} v = \langle\, \Delta\psi, i\psi \,\rangle$. Then $\dot{\psi} = \frac{i}{2}\Delta\psi$ automatically implies $\dot{\rho} = 0$.
Evolve ψ over the time intervall $[t, t+\delta]$ according to $\dot{\psi} = \frac{i}{2}\Delta\psi$. Afterwards repair constraints: $\psi \rightsquigarrow \frac{\psi}{|\psi|}$.
To repair the constraint $\operatorname{div} v = 0$ we replace ψ by $e^{iq}\psi$ with $q \in C^\infty(M)$ and define the new velocity by

$$\begin{aligned} \tilde{\eta} &= \langle\, d(e^{iq}\psi), ie^{iq}\psi \,\rangle = \langle\, idq\psi + d\psi, i\psi \,\rangle \\ &= \eta + dq \end{aligned}$$

therefore

$$\tilde{v} = v + \operatorname{grad} q \qquad \text{and} \qquad \operatorname{div}\, \tilde{v} = \operatorname{div}\, v + \Delta q$$

So div $\tilde{v} = 0$ if $\Delta q = -\text{div}\ v$.
In case of $\partial M = \emptyset$ this determines q up to a constant.
So the final ψ is determined up to $\psi \mapsto e^{i\alpha}\psi$, $\alpha \in \mathbb{R}$. Indeed this is exactly pressure projection.

In praxis we use $\psi : M \to \mathbb{S}^3 \subset \mathbb{C}^2$ and define the *Hopf map*

$$\begin{aligned} \pi : \mathbb{S}^3 &\to \mathbb{S}^2 \\ \psi &\mapsto \begin{pmatrix} \bar{\psi}^t \sigma_1 \psi \\ \bar{\psi}^t \sigma_2 \psi \\ \bar{\psi}^t \sigma_3 \psi \end{pmatrix} = \begin{pmatrix} \bar{\psi}_1 \psi_2 + \bar{\psi}_2 \psi_1 \\ \frac{1}{i} \bar{\psi}_1 \psi_2 - \bar{\psi}_2 \\ |\psi_1|^2 + |\psi_2|^2 \end{pmatrix} =: s \in \mathbb{R}^3 \end{aligned} \tag{175}$$

where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{176}$$

are the *Pauli matrices.*
Then the vorticity 2-form $\omega = \iota_{\mathbf{w}} \det$ where $\mathbf{w} = \operatorname{curl} v$ or simply $\omega = d\eta$ is given by

$$\omega = \frac{1}{2} s^* dA_{\mathbb{S}^2} \tag{177}$$