Prof. Dr. Alexander I. Bobenko
Jan Techter

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/GeometryII/

Exercise Sheet 01

Exercise 1: 3D system with fields on edges. (4 pts)

Consider the following geometric system on $\mathbb{Z}^m$. The fields $x_i : \mathbb{Z}^m \to \mathbb{R}^N$ are attached to the edges $(u, u + e_i)$ parallel to the coordinate axes $\mathcal{B}_i$; see Figure 1.

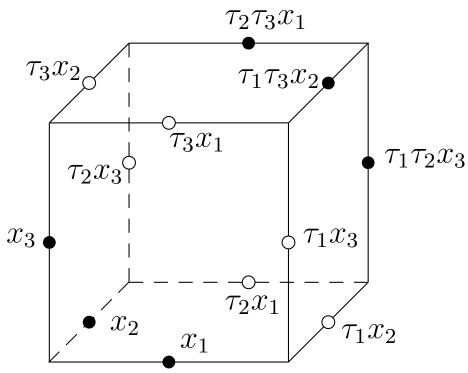

Figure 1: 3D system on an elementary cube: fields on edges.

For any elementary quadrilateral of $\mathbb{Z}^m$, it is required that the four points attached to its four edges be collinear, that is, for any $u \in \mathbb{Z}^m$ and for any $1 \leq i \neq j \leq m$, the four points x_i, x_j, $\tau_i x_j$ and $\tau_j x_i$ lie on a common straight line. (This forces the 12 points corresponding to the edges of any elementary 3D cube to be coplanar; see Figure 2.)

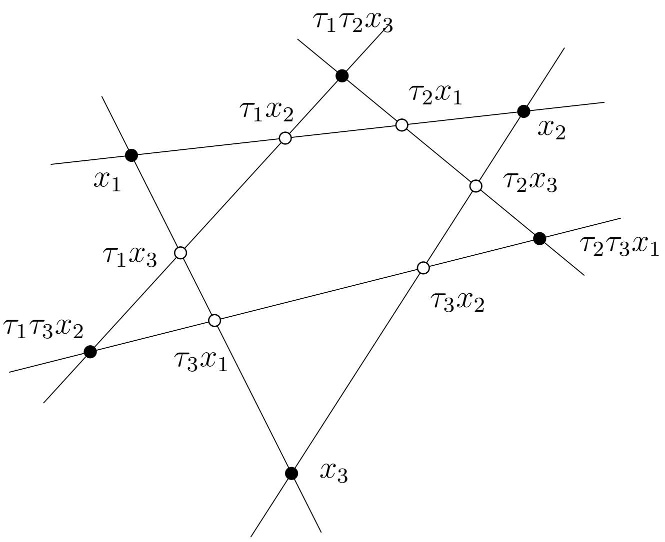

Figure 2: Geometry of the elementary cube in Figure 1.

Show that this is a 3D system with fields on edges, in the sense that the fields on six suitable chosen edges of a cube determine the other six fields uniquely.

Exercise 2: Intersection of Q-nets with a plane. (4 pts)

Given a Q-net $f : \mathbb{Z}^m \to \mathbb{R}^{N+1}$ and a generic hyperplane Π in $\mathbb{R}^{N+1}$. Intersect the edge lines of the Q-net with Π. Show that the resulting points $x_i = (ff_i) \cap \Pi$ build in Π a geometric system of Exercise 1. Describe the reverse construction. Show that all the points $x \in \Pi$ jointly with arbitrarily chosen $f \upharpoonright_{\mathcal{B}_i}$ on the coordinate axes determine the corresponding Q-net f uniquely. How does that imply the 4D consistency of the system of Exercise 1?

Exercise 3: Q-nets in homogeneous coordinates. (4 pts)

If the ambient space $\mathbb{R}^N$ of a Q-net is interpreted as an affine part of $\mathbb{R}P^N = P(\mathbb{R}^{N+1})$, then an arbitrary lift $\hat{f} = \rho(f, 1) \in \mathbb{R}^{N+1}$ of a Q-net f to the space of homogeneous coordinates $\mathbb{R}^{N+1}$ is characterized by the following condition: for every $u \in \mathbb{Z}^m$ and for every pair $i \neq j$, the four elements $\hat{f}$, $\tau_i \hat{f}$, $\tau_j \hat{f}$, and $\tau_i \tau_j \hat{f}$ are linearly dependent (span a three-dimensional vector subspace):

$$\tau_i \tau_j \hat{f} = \alpha_{ji} \tau_i \hat{f} + \alpha_{ij} \tau_j \hat{f} + \beta_{ij} \hat{f} \tag{1}$$

for some real $\alpha_{ij}, \alpha_{ji}, \beta_{ij}$.

To prove this statement proceed as follows: Let the function $\tilde{f} : \mathbb{Z}^m \to \mathbb{R}^N$ satisfy

$$\tau_i \tau_j \tilde{f} = \alpha_{ji} \tau_i \tilde{f} + \alpha_{ij} \tau_j \tilde{f} + \beta_{ij} \tilde{f} \tag{2}$$

Assume that $\tilde{f}$ is generic in the sense that for each $u \in \mathbb{Z}^m$ and for each pair of indices the three vectors $\tilde{f}$, $\tau_i \tilde{f}$, $\tau_j \tilde{f}$ are linearly independent. Show that the net $f = \frac{1}{\rho} \tilde{f}$ with a scalar function $\rho : \mathbb{Z}^m \to \mathbb{R} \setminus \{0\}$ is a Q-net in $\mathbb{R}^N$, if and only if the function ρ satisfies the same equation as $\tilde{f}$, i.e. if

$$\tau_i \tau_j \rho = \alpha_{ji} \tau_i \rho + \alpha_{ij} \tau_j \rho + \beta_{ij} \rho \tag{3}$$

Due Thursday, May 04, before the lecture.