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Geometry II - SS 2017

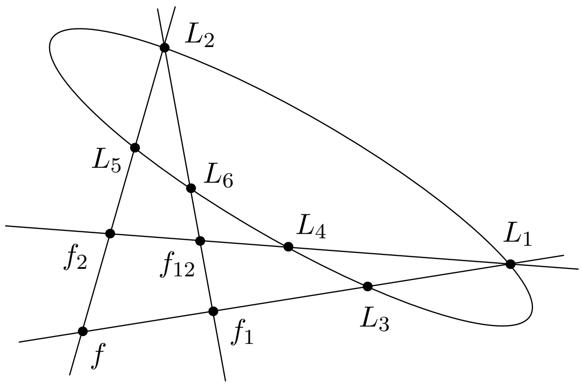
http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/GeometryII/

Exercise Sheet 04

Exercise 1: Koenigs conic. (4 pts)

Consider a planar quadrilateral $(ff_1f_{12}f_2)$ and denote the two intersection points by $L_1 = (ff_1) \cap (f_2f_{12})$ and $L_2 = (ff_2) \cap (f_1f_{12})$. Consider four further points $L_3 \in (ff_1)$, $L_4 \in (f_2f_{12})$, $L_5 \in (ff_2)$, $L_6 \in (f_1f_{12})$.
Show that the six points $L_1, \ldots, L_6$ belong to a conic, if and only if the following relation is satisfied:

$$cr(f, L_1, f_1, L_3) \cdot cr(f_2, L_4, f_{12}, L_1) = cr(f, L_2, f_2, L_5) \cdot cr(f_1, L_6, f_{12}, L_2).$$



Hint: Remember, a conic in a plane can be defined as the set of points whose homogeneous coordinates $[x_0, x_1, x_2]^T$ satisfy the equation $[x_0, x_1, x_2] \cdot \mathcal{Q} \cdot [x_0, x_1, x_2]^T = 0$, where $\mathcal{Q}$ is a symmetric 3×3 matrix.
Choose suitable homogeneous coordinates, for ex. $f = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $f_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $f_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $f_{12} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Exercise 2: Hirota-Miwa equation. (4 pts)

Consider an m-dimensional T-net with coefficients a_{ij}, which therefore satisfy the star-triangle equations

$$\frac{\tau_1 a_{23}}{a_{23}} = \frac{\tau_2 a_{31}}{a_{31}} = \frac{\tau_3 a_{12}}{a_{12}} = -\frac{1}{a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12}}.$$

Show that one can introduce a real-valued function σ defined on the vertices of $\mathbb{Z}^m$, such that

$$a_{ij} = -a_{ji} = \frac{(\tau_i\sigma)(\tau_j\sigma)}{(\tau_i\tau_j\sigma)\sigma}, \quad i > j. \tag{1}$$

Prove further that σ satisfies the equation

$$\sigma(\tau_i\tau_j\tau_k\sigma) = (\tau_i\sigma)(\tau_j\tau_k\sigma) - (\tau_j\sigma)(\tau_i\tau_k\sigma) + (\tau_k\sigma)(\tau_i\tau_j\sigma), \quad i < j < k$$

(the so called *Hirota-Miwa equation*, or *discrete BKP equation*).

Hint: Equation (1) can be interpreted as a 2D-system for the values of σ, thus it remains to show 3D- and mD-consistency ...

Exercise 3: Cox's chain of theorems. (4 pts)

Cox's chain of theorems is an infinite sequence, started by the following three theorems and then continued analogously (all objects are contained in $\mathbb{R}^3$):

COX'S FIRST THEOREM. Let Π_1, Π_2, Π_3, Π_4 be four planes in general position through a point f. Let f_{ij} be an arbitrary point on the line $\Pi_i \cap \Pi_j$. Let Π_{ijk} denote the plane through f_{ij}, f_{ik}, f_{jk}. Then the four planes Π_{123}, Π_{124}, Π_{134}, Π_{234} all pass through one point f_{1234}.

COX'S SECOND THEOREM. Let Π_1, ..., Π_5 be five planes in general position through f. Then the five points f_{1234}, f_{1235}, f_{1245}, f_{1345}, f_{2345} all lie in one plane Π_{12345}.

COX'S THIRD THEOREM. Let Π_1, ..., Π_6 be six planes in general position through f. Then the six planes Π_{12345}, Π_{12346}, Π_{12356}, Π_{12456}, Π_{13456}, Π_{23456} all pass through one point f_{123456}.

Prove Cox's chain of theorems.

Due Monday, May 29, before the lecture.