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Geometry II - SS 2017

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/GeometryII/

Exercise Sheet 05

Exercise 1: Möbius transformations in the complex plane. (3 pts)

A Möbius transformation on $\mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}} \cong \mathbb{C}\mathrm{P}^1$ is defined by

$$m(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$.

(i) Show that the cross-ratio

$$q(z_1, z_2, z_3, z_4) = \frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_3 - z_4}{z_4 - z_1}$$

of four points (z_1, z_2, z_3, z_4) in $\hat{\mathbb{C}}$ is invariant under Möbius transformations, that is $q(z_1, z_2, z_3, z_4) = q(m(z_1), m(z_2), m(z_3), m(z_4))$.

(ii) Show that every Möbius transformation can be expressed as a suitable composition of the following maps: dilatation and rotation about 0 ($w \mapsto \alpha w$ for $\alpha \in \mathbb{C} \setminus \{0\}$), translation ($w \mapsto w + \beta$ for $\beta \in \mathbb{C}$) and the inversion $w \mapsto \frac{1}{w}$.

(iii) Let $\mathcal{C}$ denote the set of all circles and straight lines in $\mathbb{C}$. Use part (ii) and show that $m(\mathcal{C}) = \mathcal{C}$, that is, each circle or straight line is mapped to a circle or straight line.

Hint: Circles in $\mathbb{C}$ are given by the set $\{z \in \mathbb{C} \mid |z-c|^2 - r^2 = |z|^2 - 2\mathrm{Re}(z\bar{c}) + |c|^2 - r^2 = 0\}$ for some $c \in \mathbb{C}$, $r > 0$.

Exercise 2: Cross-ratio of circular qudrilaterals. (3 pts)

Prove that a quadrilateral (f_1, f_2, f_3, f_4) in $\mathbb{C}$ is circular, if and only if its cross-ratio is real:

$$q(f_1, f_2, f_3, f_4) = \frac{f_1 - f_2}{f_2 - f_3} \cdot \frac{f_3 - f_4}{f_4 - f_1} \in \mathbb{R}.$$

Now consider a circular quadrilateral (f_1, f_2, f_3, f_4) in $\mathbb{C}$. Prove that it is embedded (i.e., its opposite edges do not intersect), if and only if its (real) cross-ratio is negative:

$$q(f_1, f_2, f_3, f_4) < 0.$$

Hint: Apply a suitable Möbius transformation from exercise 1.

Exercise 3: Cross-ratios on a topological sphere. (3 pts)

Consider an elementary hexahedron of a circular net. Prove that the cross-ratios $q_1, \dots, q_6$ of the six circular faces (after choosing a certain permutation of the vertices for each face) satisfy that the product $\prod_{i=1}^{6} q_i$ is equal to 1.

Now generalize this statement for a circular closed oriented quad-surface which is a topological sphere, i.e., a cell decomposition of the sphere with all faces being quadrilaterals inscribed in circles.

Exercise 4: Generalization of Miquel's theorem. (3 pts)

Prove the following "spatial generalization" of Miquel's theorem:

Consider a tetrahedron with the vertices f_1, f_2, f_3, f_4, and choose a point f_{ij} on each line $(f_i f_j)$. Then the four spheres $\tau_i S_{jkl}$ through $(f_i, f_{ij}, f_{ik}, f_{il})$ intersect at one point f_{1234}.

Hint: Use the 4D consistency of circular nets.

Due Thursday, June 01, before the lecture.