TECHNISCHE UNIVERSITÄT BERLIN
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Geometry II - SS 2017

http://www3.math.tu-berlin.de/geometrie/Lehre/SS17/GeometryII/

Exercise Sheet 07

Exercise 1: Conformal squares. (6 pts)

Definition. The image of a square in $\mathbb{C}$ under a Möbius transformation is a *conformal square*.

(i) Prove that a quadrilateral is a conformal square if and only if its cross-ratio is $q = -1$.

(ii) Consider a conformal square in $\mathbb{C}$ with the sides a, b, c, d (so $a + b + c + d = 0$). Show that there exists a quadrilateral with the edges

$$a^* = \frac{1}{\bar{a}} = \frac{a}{|a|^2}, \quad b^* = -\frac{1}{\bar{b}} = -\frac{b}{|b|^2}, \quad c^* = \frac{1}{\bar{c}} = \frac{c}{|c|^2}, \quad d^* = -\frac{1}{\bar{d}} = -\frac{d}{|d|^2},$$

that is, $a^* + b^* + c^* + d^* = 0$, and that this quadrilateral is dual to the original one.

(iii) The same statement as in (ii) holds, if the original quadrilateral is circular, with the real cross-ratio

$$q = q(a, b, c, d) = \frac{ac}{bd} = \frac{\alpha}{\beta}, \qquad \alpha, \beta \in \mathbb{R}^*,$$

and

$$a^* = \alpha \frac{1}{\bar{a}} = \alpha \frac{a}{|a|^2}, \quad b^* = \beta \frac{1}{\bar{b}} = \beta \frac{b}{|b|^2}, \quad c^* = \alpha \frac{1}{\bar{c}} = \alpha \frac{c}{|c|^2}, \quad d^* = \beta \frac{1}{\bar{d}} = \beta \frac{d}{|d|^2}.$$

Exercise 2: Christoffel transformation of discrete isothermic nets. (4 pts)

Let a Q-net $f : \mathbb{Z}^m \to \mathbb{R}^N$ be given. Assume that there exists an edge labelling α_i such that the discrete one-form

$$\delta_i f^* = \alpha_i \frac{\delta_i f}{\|\delta_i f\|^2}$$

is exact. Prove that f is discrete isothermic, with cross-ratios $q(f, f_i, f_{ij}, f_j) = \frac{\alpha_i}{\alpha_j}$.

Hint: Identify the plane of the quadrilateral (f, f_i, f_{ij}, f_j) with $\mathbb{C}$ and show that the exactness condition is equivalent to the factorization of the cross-ratios.

Exercise 3: Embedding of discrete isothermic nets. (2 pts)

Show that a discrete isothermic net $f : \mathbb{Z}^m \to \mathbb{R}^N$ of dimension $m \geq 3$ always contains quadrilaterals which are not embedded.

Due Monday, June 26, before the lecture.