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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS18/ComplexAnalysis/

Exercise Sheet 4 & 5

Due before the tutorials on Monday, May 28, 2018.

Exercise 1: Holomorphic functions and mirroring. (3 pts)

Let $U \subseteq \mathbb{C}$ be a domain, and $f : U \to \mathbb{C}$ be holomorphic. Let $\overline{U} := \{\overline{z} \mid z \in U\}$. Are the following true or false? Prove your answer.
1. $g(z) = f(\overline{z})$ is holomorphic in $\overline{U}$.
2. $g(z) = \overline{f(z)}$ is holomorphic in U.
3. $g(z) = \overline{f(\overline{z})}$ is holomorphic in $\overline{U}$.

Exercise 2: Identity theorem. (6 pts)

1. Show that there is no entire function $f : \mathbb{C} \to \mathbb{C}$ with
$$f\left(\frac{1}{n}\right) = \frac{n}{2n-1}$$
for all $n \in \mathbb{N}$.

2. Let f be an entire function such that $f(\mathbb{R}) \subseteq \mathbb{R}$ and $f(i\mathbb{R}) \subseteq i\mathbb{R}$. Show that f is odd, that is, $f(-z) = -f(z)$.

Exercise 3: Maximum principle. (4 pts)

Let f be holomorphic on a domain U, and let $K \subset U$ be connected and compact with non-empty interior. Prove that if $|f|$ is constant on ∂K, then f has a zero in $\mathrm{int}(K)$ or is constant.

Exercise 4: Schwarz' lemma. (4 pts)

Let $f : \mathbb{D} \to U \subset \mathbb{C}$ be biholomorphic. Show that
$$|f'(0)| \geq \mathrm{dist}(f(0), \partial U).$$
Remark: $\mathrm{dist}(p, \partial U) := \sup\{R > 0 \mid B_R(p) \subset U\}$, where $B_R(p)$ is a disk of radius R around $p \in U$.

Exercise 5: Power series theorem. (3 pts)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and let $C > 0$, $n \in \mathbb{N}$ and $R > 0$ such that
$$|f(z)| \leq C|z|^n$$
for all z in $\mathbb{C}$ with $|z| \geq R$. Prove that f is a polynomial of degree at most n.

Exercise 6: Liouville's theorem. (4 pts)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then f is constant in all of the following cases:
1. $\mathbb{C} \setminus f(\mathbb{C})$ contains a non-empty open disc $D_r(a) = \{z \in \mathbb{C} \mid |z - a| < r\}$.
2. $\overline{f(\mathbb{C})} \neq \mathbb{C}$.

The first condition means that the values of an entire function CANNOT be dense in $\mathbb{C}$.

Hint for 1.: Consider the function $\frac{1}{f(z)-a}$.

Exercise 7: Biholomorphic functions. (10 pts)

Let G be a domain and $f : G \to \mathbb{C}$ a holomorphic map. Let $M := \{z \in G \mid f(z) \in \mathbb{R}\}$, and $z_0 \in M$. Prove the following statements:

1. If $f'(z_0) \neq 0$ then there exists a neigbourhood V of z_0 in G such that $M \cap V$ is the image of a curve $\gamma : (0,1) \to G$.
2. If $f'(z_0) = 0$ and the function $f - f(z_0)$ has a zero at z_0 of order $= n \geq 2$ then there exists a neighbourhood V of z_0 such that $M \cap V$ is the union of n curves. Furthermore, the intersection angles of these curves at z_0 are multiples of $\frac{2\pi}{n}$.

Calculate and draw the set M for the following maps: exp, $z \mapsto z^3$, and $\sin z$.