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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS18/ComplexAnalysis/

Exercise Sheet 6

Due before the tutorials on Monday, June 4, 2018.

Exercise 1: Singularities. (4 pts)

Determine and classify the singularities of the following functions:

(i) $f_1(z) = \dfrac{z^2-1}{z^3-1}$ (ii) $f_2(z) = \dfrac{z^2+iz}{(z^2+1)^3}\cosh(\pi z/2)$ (iii) $f_3(z) = \dfrac{z}{e^z-1}$ (iv) $f_4(z) = z\sin\left(z+\dfrac{1}{z}\right)$

Exercise 2: Laurent series. (4 pts)

Let $f : \mathbb{C} \setminus \{1, -2i\} \to \mathbb{C}$,

$$f(z) = \frac{5}{z^2 + (2i-1)z - 2i}.$$

1. Determine the Laurent series for f in the annuli

$$A_1 = \{z \in \mathbb{C} \mid 1 < |z| < 2\},$$
$$A_2 = \left\{z \in \mathbb{C} \mid 0 < |z+2i| < \sqrt{5}\right\},$$
$$A_3 = \left\{z \in \mathbb{C} \mid \sqrt{5} < |z-1|\right\}.$$

2. Compute the following integrals using Part 1:

$$\int_{|z|=\frac{3}{2}} \frac{1}{z^2+(2i-1)z-2i}dz, \qquad \int_{|z+2i|=1} \frac{1}{(z+2i)^2(z^2+(2i-1)z-2i)}dz.$$

Exercise 3: Zeros and poles. (4 pts)

Let f, g and h be holomorphic and let z_0 be a pole of order ℓ of f, a pole of order m of g, and a zero of order n of h. Determine the type of the singularity in z_0 of the maps

$$f+g, \quad f+h, \quad fg, \quad fh, \quad \frac{f}{g}, \quad \frac{f}{h}, \quad \text{and} \quad \frac{h}{f}.$$

Use this to prove that the space $\mathcal{M}(G) := \{f : G \to \mathbb{C} \mid f \text{ meromorphic}\}$ is a field, where $G \subseteq \mathbb{C}$ is a domain.

Exercise 4: Holomorphic automorphisms of $\mathbb{C}$. (4 pts)

1. Let $U \in \mathbb{C}$ be open and let $f : U \to \mathbb{C}$ be holomorphic and injective. Let z_0 be an isolated singularity of f. Show that z_0 cannot be essential.
2. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be holomorphic and injective. Show that $z_0 = 0$ is either a removable singularity or a pole of first order.
3. Let $g : \mathbb{C} \to \mathbb{C}$ be holomorphic and bijective. Show that $f(z) = g(\frac{1}{z})$ cannot have a removable singularity in $z_0 = 0$. (Thus, it has a pole of first order in z_0.)
4. Determine all biholomorphic functions $g : \mathbb{C} \to \mathbb{C}$. (These are called holomorphic automorphisms of $\mathbb{C}$.)