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Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS18/ComplexAnalysis/

Exercise Sheet 7

Due before the tutorials on Monday, June 11, 2018.

Exercise 1: Compactness of $\hat{\mathbb{C}}$. (4 pts)

1. Let $z_1, z_2 \in \hat{\mathbb{C}}$. Show that

$$d_C(z_1, z_2) = \begin{cases} 2\frac{|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|)^2}}, & \text{if } z_1, z_2 \in \mathbb{C}, \\ \frac{2}{\sqrt{1+|z_1|^2}}, & \text{if } z_1 \in \mathbb{C}, z_2 = \infty. \end{cases}$$

2. Show that the Riemannian sphere $\hat{\mathbb{C}}$ is compact with respect to the chordal metric d_C.

Exercise 2: Degrees of meromorphic functions. (4 pts)

Let $f, g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Are the following true or false? Prove your answer.

1. f is bijective and meromorphic if and only if f is a rational function with degree 1.
2. If f is rational and the degree of f is $d \geq 1$, then f' is rational with degree $d - 1$.
3. If f, g are rational with degrees d_f, d_g respectively, then $f \circ g$ is rational with degree $d_f d_g$.

Exercise 3: Singularities at infinity. (4 pts)

Let U be a domain in $\mathbb{C}$ such that $\mathbb{C} \setminus U$ is compact. Let $f : U \to \mathbb{C}$ be holomorphic.
Then $z_0 = \infty$ is called a *removable singularity/pole of order m/essential singularity* of f, if the function $g(\zeta) = f(\frac{1}{\zeta})$ has a removable singularity/pole of order m/essential singularity in $\zeta = 0$.

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that f has a removable singularity in $z = \infty$ if and only if f is constant.
2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that f has a pole of order m in $z = \infty$ if and only if f is a polynomial of order m.
3. Let $f, g : U \to \mathbb{C}$ be holomorphic with removable singularities in $z = \infty$. Show that $f = g$ if $f(k) = g(k)$ for all $k \in U \cap \mathbb{Z}$.

Exercise 4: Stereographic projection. (4 pts)

Let $\sigma : \mathbf{S}^2 \to \hat{\mathbb{C}}$ be the stereographic projection. Every $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ can be uniquely identified with a map $\tilde{f} : \mathbf{S}^2 \to \mathbf{S}^2$, given by $\tilde{f} = \sigma^{-1} \circ f \circ \sigma$. Determine $\tilde{f}_i$ for

(i) $f_1(z) = \frac{1}{\bar{z}}$ (ii) $f_2(z) = -z$ (iii) $f_3(z) = z^2$

and describe the action of $\tilde{f}_i$ on $\mathbf{S}^2$. [1]

[1] Hint: For (iii), you may want to use *cylindrical coordinates* $(x_1, x_2, x_3) = (r\cos\varphi, r\sin\varphi, x_3)$.