Technische Universität Berlin
Institut für Mathematik

Dr. Ulrike Bücking
Hana Kouřimská

# Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS18/ComplexAnalysis/

## Exercise Sheet 8

Due before the tutorials on Monday, June 18, 2018.

**Exercise 1: Special Möbius transformations, part I** (4 pts)

There are six permutations $\sigma$ of the triple $(0, 1, \infty)$. Each of them yields a Möbius transformation $f_\sigma$ determined by $f_\sigma(p) = \sigma(p)$ for $p \in (0, 1, \infty)$. Find these explicitly (meaning, write down an explicit formula $f(z) = \frac{az+b}{cz+d}$ for each transformation).

**Exercise 2: Special Möbius transformations, part II.** (6 pts)

Let $f$ be a Möbius transformation that maps the unit disc $\mathbb{D}$ to itself bijectively. Show that $f$ then fulfills the following formulas:

1. $f(z) = \frac{az+\bar{b}}{bz+\bar{a}}, \quad a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1,$
2. $f(z) = e^{i\varphi} \frac{z-z_0}{1-\overline{z_0}z}, \quad z_0 \in \mathbb{D}, \varphi \in \mathbb{R}.$

*Hint:* Use the following fact from the lecture: All Möbius transformations $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $f(\mathbb{H}) = \mathbb{H}$ are of the form $f(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{R}$, where $\mathbb{H}$ denotes the upper halfplane $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$.

**Exercise 3: Fixed points of Möbius transformations** (6 pts)

1. Let $g_1$ be a Möbius transformation having $\infty$ as only fixed point. Show that there exists $r \in \mathbb{C} \setminus \{0\}$ such that $g_1(z) = z + r$ for all $z$.
2. Let $f_1$ be a Möbius transformation with only one fixed point in $\hat{\mathbb{C}}$. Show that there exists $r \in \mathbb{C} \setminus \{0\}$ and a Möbius transformation $\tau_1$ such that $\tau_1 \circ f_1 \circ \tau_1^{-1}(z) = z + r$.
3. Let $g_2$ be a Möbius transformation with fixed points 0 and $\infty$. Show that there exists $s \in \mathbb{C} \setminus \{0, 1\}$ such that $g_2(z) = sz$ for all $z$.
4. Let $f_2$ be a Möbius transformation with two different fixed points in $\hat{\mathbb{C}}$. Show that there exists $s \in \mathbb{C} \setminus \{0, 1\}$ and a Möbius transformation $\tau_2$ such that $tau_2 \circ f_2 \circ \tau_2^{-1}(z) = sz$ for all $z \in \hat{\mathbb{C}}$.