Technische Universität Berlin
Institut für Mathematik

Dr. Ulrike Bücking
Hana Kouřimská

Berlin Mathematical School

Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS18/ComplexAnalysis/

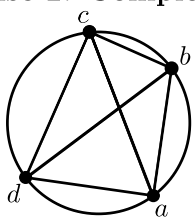
Exercise Sheet 9

Due before the tutorials on Monday, June 25, 2018.

Exercise 1: Permutations and the complex cross-ratio. (4 pts)

1. Show that if $\sigma \in S_4$ is the identity or a product of two transpositions of two disjoint pairs, that is, $(1)(2)(3)(4)$, $(12)(34)$, $(13)(24)$, or $(14)(23)$, then $\operatorname{cr}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}) = \operatorname{cr}(z_1, z_2, z_3, z_4)$.
2. Express $\operatorname{cr}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ in terms of $q \coloneqq \operatorname{cr}(z_1, z_2, z_3, z_4)$, for $\sigma \in S_4$.

Exercise 2: Complex cross-ratio and Ptolemy's theorem. (4 pts)



Let a, b, c, d be four points on a circle in cyclic order.

1. Show that $\angle acb = \angle adb$ by consideration of the cross-ratio.
2. Show that $\operatorname{cr}(z_1, z_2, z_3, z_4) + \operatorname{cr}(z_1, z_3, z_2, z_4) = 1$ for any four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$. Deduce *Ptolemy's theorem*:

$$|a-c||b-d| = |a-b||c-d| + |b-c||d-a|.$$

Exercise 3: Fixed points of Möbius transformations, part II. (4 pts)

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$, $ad - bc = 1$, that is not the identity. Show that the fixed points of f are

$$z_{\pm} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

Calculate $f'(z_{\pm})$ and deduce that $f'(z_+)\, f'(z_-) = 1$.

Exercise 4: Principal value logarithm. (4 pts)

The principal value logarithm is defined as

$$\operatorname{Log}(z) = \ln(|z|) + i\operatorname{Arg}(z), \qquad z \in \mathbb{C}\backslash\{x \in \mathbb{R} \mid x \leq 0\},$$

where $\operatorname{Arg}(re^{i\varphi}) = \varphi \in (-\pi, \pi)$.

1. Rewrite $\operatorname{Log}(z)$ in terms of x and y, where $z = x + iy$.
2. Use this expression to show that Log has derivative

$$\operatorname{Log}'(z) = \frac{1}{z}.$$

Remark: The following calculation from the lecture will NOT be accepted as valid solution:

$$e^{\operatorname{Log}(z)} = z \Rightarrow 1 = e^{\operatorname{Log}(z)} \cdot \operatorname{Log}'(z) \Rightarrow \operatorname{Log}'(z) = \frac{1}{z}.$$