Exercise 1: Permutations and the complex cross-ratio. (4 pts)
1. Show that if $\sigma \in S_4$ is the identity or a product of two transpositions of two disjoint pairs, that is, $(1)(2)(3)(4)$, $(12)(34)$, $(13)(24)$, or $(14)(23)$, then $\text{cr}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}) = \text{cr}(z_1, z_2, z_3, z_4)$.
2. Express $\text{cr}(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)})$ in terms of $q \equiv \text{cr}(z_1, z_2, z_3, z_4)$, for $\sigma \in S_4$.

Exercise 2: Complex cross-ratio and Ptolemy’s theorem. (4 pts)
1. Let $a, b, c, d$ be four points on a circle in cyclic order.
   - Show that $\angle acb = \angle adb$ by consideration of the cross-ratio.
2. Show that $\text{cr}(z_1, z_2, z_3, z_4) + \text{cr}(z_1, z_3, z_2, z_4) = 1$ for any four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$. Deduce Ptolemy’s theorem:
   \[|a - c||b - d| = |a - b||c - d| + |b - c||d - a|\]

Exercise 3: Fixed points of Möbius transformations, part II. (4 pts)
Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation $z \mapsto \frac{az + b}{cz + d}$, $ad - bc = 1$, that is not the identity. Show that the fixed points of $f$ are
\[z_\pm = \frac{(a - d) \pm \sqrt{(a + d)^2 - 4c}}{2c}.
\]
Calculate $f'(z_\pm)$ and deduce that $f'(z_+) f'(z_-) = 1$.

Exercise 4: Principal value logarithm. (4 pts)
The principal value logarithm is defined as
\[\text{Log}(z) = \ln(|z|) + i \text{Arg}(z), \quad z \in \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\},\]
where $\text{Arg}(re^{i\varphi}) = \varphi \in (-\pi, \pi)$.
1. Rewrite $\text{Log}(z)$ in terms of $x$ and $y$, where $z = x + iy$.
2. Use this expression to show that $\text{Log}$ has derivative
\[\text{Log}'(z) = \frac{1}{z}.
\]
Remark: The following calculation from the lecture will NOT be accepted as valid solution:
\[e^{\text{Log}(z)} = z \Rightarrow 1 = e^{\text{Log}(z)} \cdot \text{Log}'(z) \Rightarrow \text{Log}'(z) = \frac{1}{z}.
\]