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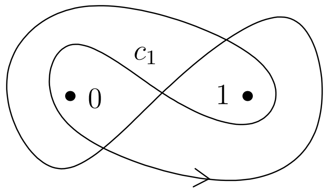
Complex Analysis I

http://www3.math.tu-berlin.de/geometrie/Lehre/SS18/ComplexAnalysis/

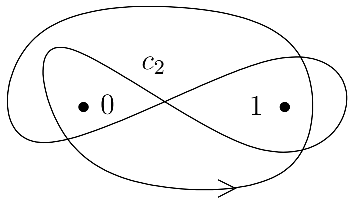
Exercise Sheet 10

Due before the tutorials on Monday, July 2, 2018.

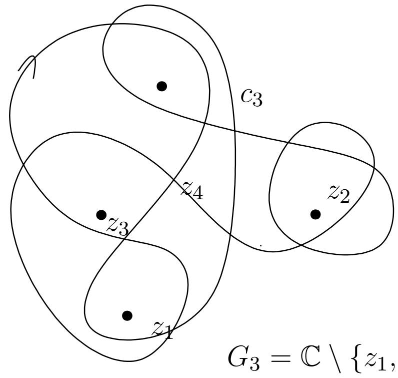
Exercise 1: Homotopy. (3 pts)



$G_1 = \mathbb{C} \setminus \{0, 1\}$



$G_2 = \mathbb{C} \setminus \{0, 1\}$



$G_3 = \mathbb{C} \setminus \{z_1, z_2, z_3, z_4\}$

1. Which of the above curves c_i are null-homotopic in the given region G_i?
2. Determine the winding number of the curves c_i around each point in the complement of G_i.

Exercise 2: Fundamental groups. (4 pts)

Let $G \subseteq \mathbb{C}, x_1, x_2 \in G$. Consider the fundamental groups $\pi_1(G, x_1), \pi_1(G, x_2)$. Show that if there is a path connecting x_1 and x_2, then $\pi_1(G, x_1)$ and $\pi_1(G, x_2)$ are isomorphic.

Exercise 3: Analytic branches of arctan. (7 pts)

Let $G = \mathbb{C}\backslash[-i, i], z_0 = 1$ and $\gamma_z : [0, 1] \to G$ a piecewise differentiable curve from z_0 to z. Let

$$F(z) = \int_{\gamma_z} \frac{1}{1 + \zeta^2}\, d\zeta.$$

Let $g : G \to \mathbb{C}, g(z) = F(z) + \pi/4$.

1. Let $\gamma_1, \gamma_2 : [0, 2\pi] \to \mathbb{C}, \gamma_1(t) = 2e^{it}, \gamma_2(t) = re^{it} + i, r \in (0, 2)$. Show that
$$\int_{\gamma_1} \frac{1}{1 + \zeta^2}\, d\zeta = 0, \qquad \int_{\gamma_2} \frac{1}{1 + \zeta^2}\, d\zeta = \pi.$$
2. Show that $F : G \to \mathbb{C}$ (and thus g) is well-defined (i.e. independent of the choice of γ_z) and holomorphic.
3. Can you increase the domain of g by analytic continuation?
Hint: Consider $g(z + \varepsilon)$ and $g(z - \varepsilon)$ for $z \in [-i, i], \varepsilon \in \mathbb{R}\backslash\{0\}, \varepsilon \to 0$, and use part 1.
4. Consider $x \in \mathbb{R}^+$. Show that
$$(\tan)'(x) = 1 + \tan^2(x) \qquad \text{and} \qquad \tan(g(x)) = x.$$
Hint: Choose γ_x as a path in $\mathbb{R}^+$ and calculate $F(x)$ explicitly via integration.
5. Show that $\tan(g(z)) = z$ for all $z \in G$. (This proves that $g = \arctan$ is one analytic branch of the inverse of the tangent.)

Exercise 4: Holomorphic logarithms. (2 pts)

Let $G \subseteq \mathbb{C}$. A *holomorphic logarithm* of a function $f : G \to \mathbb{C}$ is any function $g : G \to \mathbb{C}$ that fulfills $e^{g(z)} = f(z)$.

Find a map $f : G \to \mathbb{C}$ without zeros, that does not possess any holomorphic logarithm.