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Complex Analysis I

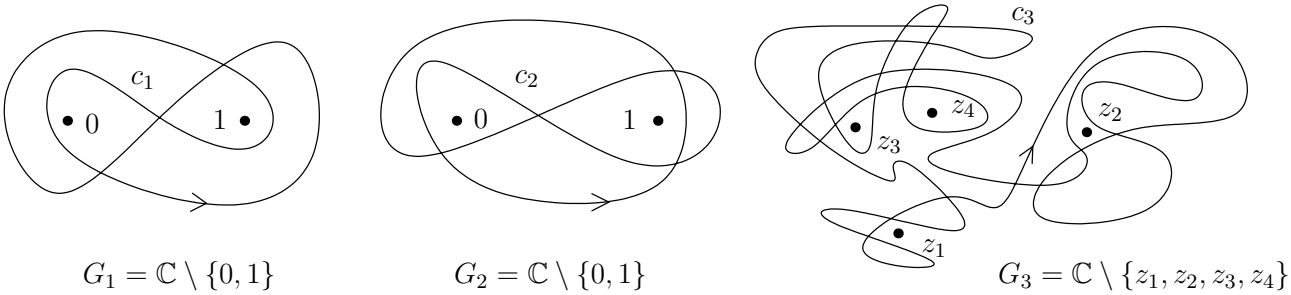
http://www3.math.tu-berlin.de/geometrie/Lehre/SS18/ComplexAnalysis/

Exercise Sheet 11

Due before the tutorials on Monday, July 9, 2018.

This is the last sheet of the course!

Exercise 1: Homology. (4 pts)



$G_1 = \mathbb{C} \setminus \{0, 1\}$ $G_2 = \mathbb{C} \setminus \{0, 1\}$ $G_3 = \mathbb{C} \setminus \{z_1, z_2, z_3, z_4\}$

1. Which of the above curves c_i are zero-homologous in the given region G_i?
2. Let $z_1, z_2, z_3 \in \mathbb{C}$ be three different points. Construct a zero-homologous curve in $\mathbb{C} \setminus \{z_1, z_2, z_3\}$ which is not null-homotopic in $\mathbb{C} \setminus \{z_i, z_j\}$ for all $i \neq j$.

Exercise 2: Dog on a leash theorem. (4 pts)

Let $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C}$ be two closed curves, and let $z_0 \in \mathbb{C} \setminus \{\mathrm{tr}(\gamma_1), \mathrm{tr}(\gamma_2)\}$. Suppose that for all $t \in [0, 1]$ holds:

$$|\gamma_1(t) - \gamma_2(t)| < |z_0 - \gamma_1(t)|.$$

Show that $n_{\gamma_1}(z_0) = n_{\gamma_2}(z_0)$.
How does the setting of the exercise relate to a situation with a dog on a leash?

Exercise 3: Residues. (5 pts)

1. Compute $\mathrm{res}(\frac{\tan z}{z^2}, 0)$.
2. Compute $\mathrm{res}(e^{z+\frac{1}{z}}, 0)$.
3. For $w \in \mathbb{C} \setminus \mathbb{Z}$ and $f(z) = \frac{\cot(\pi z)}{(z-w)^2}$ show that

$$\mathrm{res}(f, w) = -\frac{\pi}{\sin^2(\pi w)}.$$

Exercise 4: Residue theorem. (3 pts)

Verify the following integral for $0 < p < 1$:

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 - 2p\cos\theta + p^2} = \frac{2\pi}{1 - p^2}.$$

Hint: Show that $\int_0^{2\pi} \frac{\mathrm{d}\theta}{1-2p\cos\theta+p^2} = \int_{|z|=1} \frac{i\mathrm{d}z}{(z-p)(pz-1)}$.