TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. Alexander Bobenko
Jan Techter

Complex Analysis I - SoSe 2019

http://www3.math.tu-berlin.de/geometrie/Lehre/SS19/ComplexAnalysis/

Exercise Sheet 2

1. Aufgabe (Visualization of complex functions) (5 pts)

(i) Show that
$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y.$$

(ii) Draw the images of the coordinate lines $\{x+iy \mid x = \text{const.}\}$ and $\{x+iy \mid y = \text{const.}\}$ under the complex exponential and complex sine function.

(iii) Draw the level sets of the real and imaginary part of the complex sine function.

2. Aufgabe (Cauchy-Riemann equations) (5 pts)

Let U be an open, connected subset of $\mathbb{C}$, and $f : U \to \mathbb{C}$ holomorphic.

(i) Show that if real or imaginary part of f are constant, then f is constant.

(ii) Introduce polar coordinates in the image plane, i.e. $f(x,y) = r(x,y)e^{i\theta(x,y)}$, and find the Cauchy-Riemann equations in terms of $r(x,y)$ and $\theta(x,y)$.

(iii) Show that if argument or absolute value of f are constant, then f is constant.

3. Aufgabe (Harmonic functions) (5 pts)

(i) Which of the following functions is harmonic?
$$u(x,y) = e^x, \qquad u(x,y) = e^x(x\cos y - y\sin y)$$
Where applicable, find a harmonic conjugate of u, i.e. a function $v(x,y)$ satisfying $v_x = -u_y$ and $v_y = u_x$

(ii) Find the most general form for which the polynomial
$$ax^2 + 2bxy + cy^2$$
is the real part of a holomorphic function. Construct this holomorphic function, and express it in terms of z.

4. Aufgabe (Stereographic projection) (5 pts)

Let $\sigma : \mathbb{S}^2 \to \hat{\mathbb{C}}$ be the stereographic projection. Every $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ can be uniquely identified with a map $\tilde{f} : \mathbb{S}^2 \to \mathbb{S}^2$, given by

$$\tilde{f} \coloneqq \sigma^{-1} \circ f \circ \sigma.$$

Determine $\tilde{f}$ for

(i) $f(z) = \bar{z}$, (ii) $f(z) = \frac{1}{\bar{z}}$, (iii) $f(z) = -z$, (iv) $f(z) = \frac{1}{z}$, (v) $f(z) = -\frac{1}{\bar{z}}$

and describe the action of $\tilde{f}$ on $\mathbb{S}^2$.

Bonus: (4 pts)

Find the most general form of f for which $\tilde{f}$ is a rotation of $\mathbb{S}^2$.

Due Thursday, May 02.