TECHNISCHE UNIVERSITÄT BERLIN
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Geometry I
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https://www3.math.tu-berlin.de/geometrie/Lehre/SS19/GeometryII/

Exercise Sheet 10

Exercise 1: Ideal Ptolemy. (6 pts)

In the hyperboloid model of the hyperbolic plane,

$$\mathbb{H}^2 = \left\{x \in \mathbb{R}^{2,1} | \langle x, x\rangle_{2,1} = -1,\ x_3 > 0\right\},$$

a *horocycle* H corresponds to a lightlike vector h in the upper light cone

$$L_+^{2,1} = \left\{x \in \mathbb{R}^{2,1} | \langle x, x\rangle_{2,1} = 0,\ x_3 > 0\right\}$$

via the relation

$$H = \{x \in \mathbb{H}^2 | \langle x, h\rangle_{2,1} = -1\}.$$

The *signed distance* $d(H_1, H_2)$ *of horocycles* $H_1,\ H_2$ is defined as the hyperbolic distance between the two horocycles along the geodesic connecting their ideal centers, taken positively if H_1, H_2 are disjoint and negatively if they intersect, see figure 1.

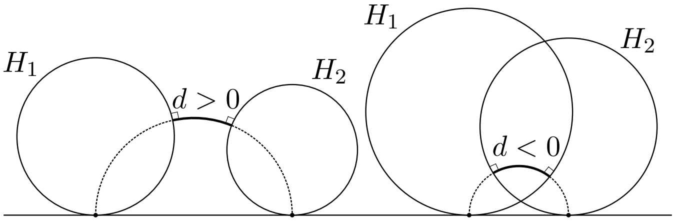

Figure 1: The signed distance of horocycles

a) Show that $\langle h_1, h_2\rangle_{2,1} = -2e^{d(H_1,H_2)}$

b) Given four distinct horocycles H_i, $i = 1, .., 4$ in cyclic order (see figure 2), prove *Ptolemy's identity for ideal quadrilaterals*:

$$e^{\frac{1}{2}(d_{13}+d_{24})} = e^{\frac{1}{2}(d_{12}+d_{34})} + e^{\frac{1}{2}(d_{23}+d_{41})},$$

where $d_{ij} = d(H_i, H_j)$.

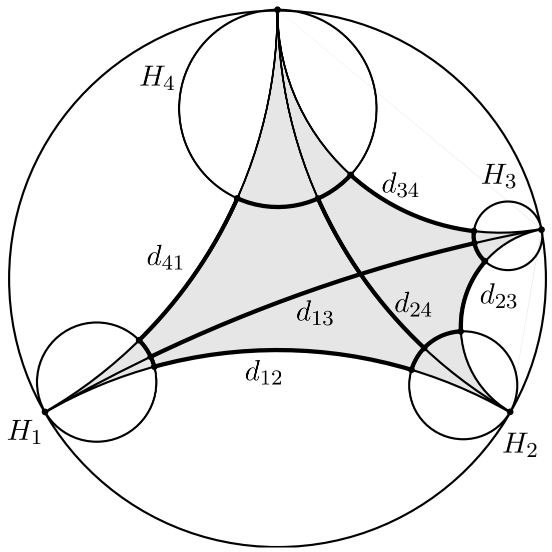

Figure 2: Ptolemy relation

Exercise 2: Tangential distance. (2 pts)

Denote by C_i a circle in the plane with center c_i and radius r_i and let

$$p_i = [c_i^T, ||c_i||^2 - r_i^2, 1, r] \in \mathcal{L}$$

be the corresponding point in the Lie quadric $\mathcal{L}$. The *tangential distance* $d(C_1, C_2)$ of two oriented circles C_1, C_2 is defined as the distance between the points of oriented contact of a common oriented tangent, if it exists.

Show that for the tangential distance holds

$$\frac{\langle p_1, p_2 \rangle_{Lie}}{\langle p_1, e_\infty \rangle_{Lie} \langle p_2, e_\infty \rangle_{Lie}} = -2d^2(C_1, C_2),$$

where $e_\infty = [0, 0, 1, 0, 0] \in \mathbb{R}P^4$.

Exercise 3: Casey's theorem. (4 pts)

Let C_i, $i = 1, .., 4$, be four non-intersecting circles in the plane that lie inside a circle C in cyclic order C_1, C_2, C_3, C_4 and are tangent to it. Denote by d_{ij} the distance between the points of tangency of a common tangent of C_i, C_j, not intersecting the line connecting their centers. Show that

$$d_{13} d_{24} = d_{12} d_{34} + d_{23} d_{41}.$$

Due: Tuesday, 25.06.2019 before the lecture