Exercise Sheet 4

Exercise 1. Inverse points. (4 pts)
Let $S$ be a hypersphere in $\mathbb{R}^n \cup \{\infty\}$ and denote by $I_S$ reflection in $S$. Two points $x, y \in \mathbb{R}^n$ are called inverse points with respect to $S$ if $x = I_S(y)$ (and, of course, $y = I_S(x)$). Let $m$ be a Möbius transformation of $\mathbb{R}^n \cup \{\infty\}$. Show:

1. If $x, y$ are inverse points with respect to $S$, then $m(x), m(y)$ are inverse points with respect to $m(S)$.
2. $x, y$ are inverse points with respect to $S$ if and only if every sphere through $x$ and $y$ is orthogonal to $S$.

Exercise 2: Möbius transformations as compositions of reflections. (4 pts)
Every Möbius transformation of $\hat{\mathbb{C}}$ can be written as the composition of at most 3 resp. 4 reflections in lines or circles. Show:

1. $z \mapsto \bar{z} + 1$ is the composition of 3 and no fewer reflections.
2. $z \mapsto mz, m \in \mathbb{C} \setminus \mathbb{R}, |z| \neq 1$ cannot be written as the composition of two reflections.

Exercise 3: Fixpoints of Möbius transformations. (4 pts)
Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation, $f \neq id$, and $(a \ b \ c \ d) \in SL(2, \mathbb{C})$ a corresponding (normalized) $2 \times 2$-matrix.

1. Show that

   \[ f \text{ parabolic } :\leftrightarrow \ a + d = \pm 2 \]

   \[ \leftrightarrow \ f \text{ has exactly one fixpoint.} \]
2. Otherwise $f$ has exactly two fixpoints and there exists a Möbius transformation $g$ and $m \in \mathbb{C} \setminus \{0, 1\}$ such that

$$(g \circ f \circ g^{-1})(z) = mz.$$ 

Show that

- $f$ elliptic $\iff a + d \in \mathbb{R}$ and $|a + d| < 2$ 
  $\iff m = e^{i\varphi} \neq 1$ for some $\varphi \in \mathbb{R}$

- $f$ hyperbolic $\iff a + d \in \mathbb{R}$ and $|a + d| > 2$ 
  $\iff m = \rho \neq 1$ for some $\rho > 0$.

Hint: Show that $m = \lambda^2$ where $\lambda$ satisfies $\lambda + \frac{1}{\lambda} = a + d$.

Due: Tuesday, 14.05.2019 before the lecture