TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

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Geometry I
Summer 2019

https://www3.math.tu-berlin.de/geometrie/Lehre/SS19/GeometryII/

Exercise Sheet 4

Exercise 1. Inverse points. (4 pts)

Let S be a hypersphere in $\mathbb{R}^n \cup \{\infty\}$ and denote by I_S reflection in S. Two points $x, y \in \mathbb{R}^n$ are called *inverse points with respect to* S if $x = I_S(y)$ (and, of course, $y = I_S(x)$). Let m be a Möbius transformation of $\mathbb{R}^n \cup \{\infty\}$. Show:

1. If x, y are inverse points with respect to S, then $m(x), m(y)$ are inverse points with respect to $m(S)$.
2. x, y are inverse points with respect to S if and only if every sphere through x and y is orthogonal to S.

Exercise 2: Möbius transformations as compositions of reflections. (4 pts)

Every Möbius transformation of $\hat{\mathbb{C}}$ can be written as the composition of at most 3 resp. 4 reflections in lines or circles. Show:

1. $z \mapsto \bar{z} + 1$ is the composition of 3 and no fewer reflections.
2. $z \mapsto mz, m \in \mathbb{C} \backslash \mathbb{R},\ |z| \neq 1$ can not be written as the composition of two reflections.

Exercise 3: Fixpoints of Möbius transformations. (4 pts)

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation, $f \neq id$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$ a corresponding (normalized) 2×2-matrix.

1. Show that

$$\begin{aligned} f \text{ parabolic } &:\Leftrightarrow\ a + d = \pm 2 \\ &\Leftrightarrow\ f \text{ has exactly one fixpoint.} \end{aligned}$$

2. Otherwise f has exactly two fixpoints and there exists a Möbius transformation g and $m \in \mathbb{C} \setminus \{0, 1\}$ such that

$$\left(g \circ f \circ g^{-1}\right)(z) = mz.$$

Show that

$$\begin{aligned} f \text{ elliptic } &:\Leftrightarrow\ a + d \in \mathbb{R} \quad \text{and} \quad |a + d| < 2 \\ &\Leftrightarrow\ m = e^{i\varphi} \neq 1 \quad \text{for some } \varphi \in \mathbb{R} \\ f \text{ hyperbolic } &:\Leftrightarrow\ a + d \in \mathbb{R} \quad \text{and} \quad |a + d| > 2 \\ &\Leftrightarrow\ m = \rho \neq 1 \quad \text{for some } \rho > 0. \end{aligned}$$

Hint: Show that $m = \lambda^2$ where λ satisfies $\lambda + \frac{1}{\lambda} = a + d$.

Due: Tuesday, 14.05.2019 before the lecture