Exercise Sheet 5

Exercise 1. Orthogonal spheres. (4 pts)
Show that there are $n + 1$ but no $n + 2$ pairwise orthogonal hyperspheres in $\mathbb{R}^n$.

Exercise 2: Orthogonal group. (4 pts)
Let $\langle \cdot, \cdot \rangle$ denote the indefinite scalar product of $\mathbb{R}^{p,q}$, and for a non-isotropic vector $w \in \mathbb{R}^{p,q}$ let the map $R_w : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ be defined by
\[
R_w(x) = x - 2\frac{\langle x, w \rangle}{\langle w, w \rangle}w.
\]
Show:
1. $R_w \in O(p, q)$.
2. If $A \in O(p, q)$, $v \in \mathbb{R}^{p,q}$ and $w = Av - v$ is non-isotropic, then $R_w(v) = Av$ and $R_w(Av) = v$.

Exercise 3: Skew symmetric bilinear forms. (4 pts)
Let $\omega$ be a skew-symmetric non-degenerate bilinear form on an $n$-dimensional real vector space $V$. This means
\begin{itemize}
  \item $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$,
  \item for each $v \in V \setminus \{0\}$ there is a $w \in v$ with $\omega(v, w) \neq 0$.
\end{itemize}
Show that there is a basis $(v_1, \ldots, v_n)$ of $V$ such that the matrix $\Omega = (\omega(v_i, v_j))_{ij} \in \mathbb{R}^{n \times n}$ is
\[
\Omega = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 1 \\
-1 & 0 & & & & \\
0 & 1 & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & 1 & & & & \\
& & & & & \\
-1 & 0 & & & & \\
\end{pmatrix}.
\]
In particular, show that $n$ is even.

Due: Tuesday, 21.05.2019 before the lecture