TECHNISCHE UNIVERSITÄT BERLIN
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Geometry I
Summer 2019

https://www3.math.tu-berlin.de/geometrie/Lehre/SS19/GeometryII/

Exercise Sheet 7

Exercise 1: Möbius geometric pencils of circles. (4 pts)

In the projective model of Möbius geometry, circles in $\widehat{\mathbb{R}^2}$ correspond to points outside the sphere

$$\mathbb{S}^2 = \left\{[x] \in \mathbb{R}\mathrm{P}^3 | \langle x, x\rangle_{3,1} = 0\right\}.$$

A line $l = P(U)$ in $\mathbb{R}P^3$ corresponds to a one-parameter family of circles called a *pencil of circles*.

Show: If the restriction of $\langle \cdot, \cdot \rangle_{3,1}$ to U has signature

1. $(++)$ then all circles in the pencil share two common points;
2. $(+-)$ then any two circles in the pencil are disjoint;
3. $(+0)$ then the circles in the pencil are tangent to each other at a common point.

In these cases, the pencil of circles is called 1.hyperbolic, 2.elliptic, 3.parabolic.

Exercise 2: Euclidean motions in the projective model. (4 pts)

Let $\sigma : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3 \subset \mathbb{R}\mathrm{P}^3$ be stereographic projection in the north pole, given in homogeneous coordinates of $\mathbb{R}\mathrm{P}^3$ by

$$\sigma : u = (u_1, u_2) \mapsto [2u_1, 2u_2, ||u||^2 - 1, ||u||^2 + 1]^T.$$

By this correspondence, points in $\mathbb{R}^2$ correspond to points in $\mathbb{S}^2$ and every Möbius transformation of the plane corresponds to an element of $PO(3,1)$.

Let $A \in O(2),\ b \in \mathbb{R}^2$ and consider the Euclidean motion $f_{A,b}(u) = Au + b$.

1. Find $M_{A,b} \in O(3,1)$ such that $f_{A,b}$ corresponds to $M_{A,b}$ in the projective model. (Verify that $M_{A,b} \in O(3,1)$.)
2. Show that 1 is an eigenvalue of $M_{A,b}$ to the eigenvector $[0,0,1,1]^T$.

Hint: $\langle Au, b\rangle = \langle u, A^{-1}b\rangle$.

Exercise 3. Orthogonal complement of a line. (4 pts)

Consider the quadric

$$Q = \{[y] \in \mathbb{R}P^{n+2} | y_1^2 + ... + y_n^2 + y_{n+1}^2 - y_{n+2}^2 - y_{n+3}^2 = 0\}.$$

Let l be a line contained in Q and let $l^\perp$ be the polar n-plane of l. Show the following:

1. Q contains projective lines but no k-planes for $k > 1$.
2. $l^\perp \cap Q = l$.
3. If $[x] \in l^\perp$ and $[x] \notin l$, then x is spacelike.

Due: Tuesday, 04.06.2019 before the lecture