Exercise Sheet 8

Exercise 1: Lie transformations pt.1 (4 pts)

a) Show that there is a Lie transformation that maps each oriented sphere with center $c \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$ to the oriented sphere with center $c$ and signed radius $r + h$, where $h$ is a fixed real number.

b) What is the image of the oriented plane with unit normal vector $n \in S^{n-1}$ and equation

$$(n, u) - d = 0$$

under the same Lie transformation?

Exercise 2: Lie transformations pt.2 (4 pts)

Let

$$Q = \{ [y] \in \mathbb{R}P^{n+2} | y_1^2 + \ldots + y_n^2 + y_{n+1}^2 - y_{n+2}^2 - y_{n+3}^2 = 0 \}$$

be the Lie quadric and denote by $q$ the corresponding symmetric bilinear form.

a) Let $[p_1], [p_2], [p_3] \in Q$. Show that

$$\sigma(p_1, p_2, p_3) = \text{sign}(q(p_1, p_2)q(p_2, p_3)q(p_3, p_1))$$

is invariant under Lie transformations.

b) Consider two triples of points in the Lie quadric, $[p_1], [p_2], [p_3]$ and $[q_1], [q_2], [q_3]$. Assume that all $[p_i], i = 1, 2, 3$, represent points in $\mathbb{R}^n$ and that $[q_3]$ represents an oriented hyperplane in $\mathbb{R}^n$, and $[q_1], [q_2]$ represent points in $\mathbb{R}^n$ on different sides of the hyperplane.

Show that there is no Lie transformation mapping $[p_1], [p_2], [p_3]$ to $[q_1], [q_2], [q_3]$. 
Exercise 3. Lines in the Lie quadric and oriented contact. (4 pts)

Let

\[ Q = \{ [y] \in \mathbb{P}^{n+2} | y_1^2 + ... + y_n^2 + y_{n+1}^2 - y_{n+2}^2 - y_{n+3}^2 = 0 \} \]

be the Lie quadric and denote by \( q \) the corresponding symmetric bilinear form. Let \([p], [r] \in Q\) and denote by \( l \) the projective line spanned by \( p \) and \( r \), \( l = [tp + r], \ t \in \mathbb{R} \).

Show:

a) \( l \) is contained in \( Q \) if and only if \( q(p, r) = 0 \), i.e., \([p]\) and \([r]\) are in oriented contact.

b) If \( l \subset Q \), then the set of Lie spheres in \( \mathbb{R}^n \) corresponding to points on \( l \) is precisely the set of all Lie spheres in oriented contact with both \([p]\) and \([r]\).

c) Let \([p_1], [p_2], [p_3] \in Q\) and assume that \( p_1, p_2, p_3 \in \mathbb{R}^{n+3} \) are linearly dependent. Show that \([p_1], [p_2], [p_3]\) are pairwise in oriented contact.

Due: Tuesday, 11.06.2019 before the lecture