TECHNISCHE UNIVERSITÄT BERLIN
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Geometry I
Summer 2019

https://www3.math.tu-berlin.de/geometrie/Lehre/SS19/GeometryII/

Exercise Sheet 8

Exercise 1: Lie transformations pt.1 (4 pts)

a) Show that there is a Lie transformation that maps each oriented sphere with center $c \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$ to the oriented sphere with center c and signed radius $r + h$, where h is a fixed real number.

b) What is the image of the oriented plane with unit normal vector $n \in \mathbb{S}^{n-1}$ and equation

$$(n, u) - d = 0$$

under the same Lie transformation?

Exercise 2: Lie transformations pt.2 (4 pts)

Let

$$Q = \{[y] \in \mathbb{R}P^{n+2} | y_1^2 + ... + y_n^2 + y_{n+1}^2 - y_{n+2}^2 - y_{n+3}^2 = 0\}$$

be the Lie quadric and denote by q the corresponding symmetric bilinear form.

a) Let $[p_1], [p_2], [p_3] \in Q$. Show that

$$\sigma(p_1, p_2, p_3) = \text{sign}(q(p_1, p_2)q(p_2, p_3)q(p_3, p_1))$$

is invariant under Lie transformations.

b) Consider two triples of points in the Lie quadric, $[p_1], [p_2], [p_3]$ and $[q_1], [q_2], [q_3]$. Assume that all $[p_i], i = 1, 2, 3$, represent points in $\widehat{\mathbb{R}^n}$ and that $[q_3]$ represents an oriented hyperplane in $\widehat{\mathbb{R}^n}$, and $[q_1]$, $[q_2]$ represent points in $\widehat{\mathbb{R}^n}$ on different sides of the hyperplane.
Show that there is no Lie transformation mapping $[p_1], [p_2], [p_3]$ to $[q_1], [q_2], [q_3]$.

Exercise 3. Lines in the Lie quadric and oriented contact. (4 pts)

Let

$$Q = \{[y] \in \mathbb{R}P^{n+2} | y_1^2 + ... + y_n^2 + y_{n+1}^2 - y_{n+2}^2 - y_{n+3}^2 = 0\}$$

be the Lie quadric and denote by q the corresponding symmetric bilinear form. Let $[p], [r] \in Q$ and denote by l the projective line spanned by p and r, $l = [tp + r],\ t \in \widehat{R}$. Show:

a) l is contained in Q if and only if $q(p, r) = 0$, i.e., $[p]$ and $[r]$ are in oriented contact.

b) If $l \subset Q$, then the set of Lie spheres in $\mathbb{R}^n$ corresponding to points on l is precisely the set of all Lie spheres in oriented contact with both $[p]$ and $[r]$.

c) Let $[p_1], [p_2], [p_3] \in Q$ and assume that $p_1, p_2, p_3 \in \mathbb{R}^{n+3}$ are linearily dependent. Show that $[p_1], [p_2], [p_3]$ are pairwise in oriented contact.

Due: Tuesday, 11.06.2019 before the lecture