1 Solution Assignment 8, Exercise 2

- The polar lines of all the lines going through P lie in the polar plane of P, which is E_∞. That is, the polar line of a diameter is an ideal line. This means that as a point moves along such a line, the polar plane moves parallel to itself.

- As stated in the exercise introduce modified Plücker coordinates for a: (X_a, Z_a). These are defined from the original Plücker coordinates p_{ij} for a as $X_a = (p_{12}, p_{13}, p_{14})$ and $Z_a = (p_{34}, p_{42}, p_{23})$. In the following, a reference to $\langle X_a, X_a \rangle$ signifies the ordinary inner product of 3-space.

- Recall the matrix form for calculating the polar point of a plane from assignment 6. The ideal plane is given by the coordinates $[1, 0, 0, 0]$ (remember we use the first coordinate in this exercise as the homogeneous coordinate). Applying the polarizing matrix to this plane gives the polar point $P = (0, p_{12}, p_{13}, p_{14})$. This is the ideal point P polar to the ideal plane. A diameter is a line with this as its direction vector, that is, a diameter has coordinates (X_a, Z). Planes in $\mathbb{R}^3$ with plane coordinates $[t, p_{12}, p_{13}, p_{14}]$ for $t \in \mathbb{R}$ are exactly the planes perpendicular to all the diameters. Any two such planes determine an ideal line g. A simple calculation shows that the axial Plücker coordinates of such a line are $(X_a, \mathbf{0})$ so that point-wise Plücker coordinates are given by $(\mathbf{0}, X_a)$. The polar line of this ideal line is then the desired diameter.

- To find the coordinates of the axis, apply the result of the last part of exercise 1 to the ideal line g.

$$\begin{aligned}
\hat{g} &= \langle a, a\rangle g - 2\langle g, a\rangle a \\
&= 2\langle X_a, Z_a\rangle(\mathbf{0}, X_a) - 2\langle X_a, X_a\rangle(X_a, Z_a) \\
&\cong 2\frac{\langle X_a, Z_a\rangle}{\langle X_a, X_a\rangle}(\mathbf{0}, X_a) - 2(X_a, Z_a) \\
&= -2(X_a, Z_a - \frac{\langle X_a, Z_a\rangle}{\langle X_a, X_a\rangle}X_a) \\
&\cong (X_a, Z_a - \frac{\langle X_a, Z_a\rangle}{\langle X_a, X_a\rangle}X_a)
\end{aligned}$$

  As desired, the coordinates for the diameter has the form (X_a, Z) with $\langle X_a, Z\rangle = 0$.

- When $X_a = (1, 0, 1)$ and $Z_a = (0, 1, 1)$ then the formula yields

$$\begin{aligned}
\hat{g} &= ((1, 0, 1), (0, 1, 1) - \frac{1}{2}(1, 0, 1)) \\
&= ((1, 0, 1), (-\frac{1}{2}, 1, \frac{1}{2}))
\end{aligned}$$

- The axis has no significance outside of Euclidean space.