TECHNISCHE UNIVERSITÄT BERLIN WS2005
Fachbereich 3 - Mathematik
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14. Übung Geometrie I

($\mathbb{C}P^1$)

Web site: http://www.math.tu-berlin.de/geometrie/Lehre/WS05/Geometrie1.
Note: In many buildings there is no 13^{th} floor. In the same spirit, this is $Blatt 14$.

Übungsaufgaben

Note: This assignment sheet sometimes uses the same notation for points in $\hat{\mathbb{R}}^2$ as for points in $\mathbb{C}P^1$, under the standard identification $(x, y) \leftrightarrow (x + iy, 1)$ for finite points and similarly for the point at infinity $\infty \leftrightarrow (1, 0)$. Sometimes when the first space is meant, the name is printed in boldface to signify a vector, $\mathbf{z}$; while the same point in $\mathbb{C}P^1$ will be written in plain text, z.

1. Aufgabe

Examples

In the context of $\mathbb{C}P^1$, Möbius transformations are represented by elements of $PSL(2, \mathbb{C})$. An element $g \in PSL(2, \mathbb{C})$ can represent a linear automorphism when it acts as $z \to \frac{az+b}{cz+d}$, and as a semilinear automorphism when it acts as $z \to \frac{a\bar{z}+b}{c\bar{z}+d}$. We can assume that the determinant $ad - bc = 1$ by suitable choice of scaling factor. Sometimes it is convenient to require that $ad - bc = 1$; at other times we don't need to do so. (Similar to using homogeneous coordinates for points).

- Find a linear fractional transformation that maps
  a) $(1, i, -1) \to (0, 1, \infty)$,
  b) $(0, 1, i) \to (-1, -i, 0)$,
  c) $(-1, 1, i) \to (0, 3, \infty)$.
- Find a semilinear map which inverts in the circle $|z - i| = 2$.
- Let $g \in PSL(2, \mathbb{C})$, and C a circle. Show that if x and x^* are conjugate points with respect to inversion in C, then $g(x)$ and $g(x^*)$ are conjugate points with respect to inversion in $C' := g(C)$.
- Given $g \in PSL(2, \mathbb{C})$ and a circle C with center z and radius r, find the center and radius of $C' = g(C)$.
- Show that a fractional linear transformation g with $det(g) = 1$ that is a Euclidean rotation of the Riemann number sphere has the form $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{a} \end{pmatrix}$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. Show that this g satisfies this condition $\iff$ it leaves invariant the Hermitian form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- Given four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ show that it is always possible to find $k \in \mathbb{C}$ and $g \in PSL(2,\mathbb{C})$ such that $g : (z_1, z_2, z_3, z_4) \to (1, -1, k, -k)$.How many different solutions are there to this problem, and how are they related?

2. Aufgabe

We defined an inner product on the space of Hermitian forms via

$$\langle H, K\rangle := \frac{1}{2} tr(\hat{H}\hat{K})$$

- Show that $\hat{H}\hat{K} + \hat{K}\hat{H}$ is a multiple of the identity.
- Show that $tr(\hat{H}\hat{K}) = tr(\hat{K}\hat{H})$.
- Show that this inner product agrees with the inner product previously introduced on the space of circles, (the vector space V from Assignment Sheet 10), by

$$\langle \mathbf{x}, \mathbf{y}\rangle := -x_0y_3 - x_3y_0 + 2(x_1y_1 + x_2y_2)$$

Hausaufgaben

1. Aufgabe (3 Punkte)

- Show that the Hermitian form H for the Euclidean line

$$L := \{\mathbf{z} \in \hat{\mathbb{R}}^2 \mid \langle \mathbf{z}, \mathbf{n}\rangle = d\}$$

has the matrix form $H = \begin{pmatrix} 0 & -\mathbf{n} \\ -\overline{\mathbf{n}} & 2d \end{pmatrix}$. That is,

$$H(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = 0 \Leftrightarrow \left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] \in L$$

- Calculate the associated operator $\hat{H} : V \to V$ and show that it is the Euclidean reflection in the line L.

2. Aufgabe (3 Punkte)

Hermitian forms Find the Hermitian forms H corresponding to the point $z = 1 + 2i$, to the circle described by the equation $|z + (1 + i)| = 1$, and the line $z = 1$; and find the corresponding reflection operators $\hat{H}$ for each. You don't need to normalize the determinant of $\hat{H}$ to 1. Finally, analyse the Möbius transformation $\hat{H}_2\hat{H}_3$ with respect to its fixed points; sketch its action.

3. Aufgabe (3 Punkte)

Cross ratio

- Let z_1 and z_3 be two points of a circle C, and z_2 and z_4 are two symmetric points with respect to C (that is, z_2 and z_4 are images of one another via inversion in C). Show that $|cr(z_1, z_2, z_3, z_4)| = 1$.
- Suppose C_1 and C_2 intersect in z_1 and z_3, and $z_2 \in C_1$ and $z_4 \in C_2$ such that all four points are different. Show that $cr(z_1, z_2, z_3, z_4) = re^{i\theta}$, where θ is the angle between the two circles.

4. Aufgabe (4 Punkte)

Linear fractional transformations [Gebrochene Linearabbildungen]

- Find $g \in PSL(2, \mathbb{C})$ such that g maps the circle $|z - i| = 2$ to the real axis. How many such g are there?
- Let $ad - bc = 1$ and denote by C the circle $|cz + d| = 1$. Show that the linear transformation $w = \frac{az+b}{cz+d}$ increases lengths within C and decreases lengths outside C.
- **Optional** Given a circle C_1 in $\hat{\mathbb{R}}^2 \cong \mathbb{C}P^1$, find $A \in GL(\mathbb{C}, 2)$ such that $A(C_1) = \mathbb{R}$.

5. Aufgabe (6 Punkte)

Subgroups of $PSL(2, \mathbb{C})$

For $g \in PSL(2, \mathbb{C})$ assume $det(g) = 1$.

- Characterize the elements of $PSL(2, \mathbb{C})$ such that $g(\infty) = \infty$ and show they form a group.
- Characterize the elements of $PSL(2, \mathbb{C})$ which preserve the Euclidean distance (i.e, view $\mathbb{C}$ as the Euclidean plane). Show these elements form a group.
- Show that a fractional linear transformation g with $det(g) = 1$, that preserves the real axis has the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$.
- Show that a fractional linear transformation g with $det(g) = 1$ that preserves the unit circle $|z| = 1$ has the form $\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{a} \end{pmatrix}$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 - |\beta|^2 = 1$.

Note: it is possible to realize plane hyperbolic geometry as the interior of the unit disk differently than we did with $\mathbb{R}P^2$. This second model is the so-called Poincare, or conformal, model. In it, the hyperbolic isometries are elements of the subgroup of $PSL(2, \mathbb{C})$ which preserve the unit circle, and straight lines are arcs of circles which cut the unit circle at right angles.

6. Aufgabe (0 Punkte)

Optional exercise. Points based on thoroughness, clarity of answers.

- Characterize pencils of circles [Kreisbuscheln] from the point of view of $\mathbb{C}P^1$, using Hermitian forms to describe circles, etc.

Gesamtpunktzahl: 19