TECHNISCHE UNIVERSITÄT BERLIN WS2005
Fachbereich 3 - Mathematik
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2. Übung Geometrie I

(Projective transformations, harmonic fourth)

Web site: http://www.math.tu-berlin.de/geometrie/Lehre/WS05/Geometrie1. New: download the file *notation.pdf* which tries to clarify some notational issues related to this course.

Übungsaufgaben

1. Aufgabe

Construction of the harmonic fourth point.

1. Draw an initial line l.
2. Choose and mark three points M, N, and P on l.
3. Construct lines m, n, and p through the points M, N, P, respectively, so that the resulting four lines are in general position.
4. Construct and mark the points $A := np$, $B := pm$, $C = mn$.
5. Construct the lines $m' := MA$, $n' = NB$.
6. Construct the point $D := m'n'$.
7. Construct the line $q := CD$.
8. Construct the point $Q := ql$.

Q is called the *harmonic fourth* with respect to M, N, and P. We'll prove in class that no matter how the construction is carried out, you get the same point Q, and further that $cr(M, P, N, Q) = -1$.

2. Aufgabe

Duality Last week's assignments involving lines and points should have convinced you that in $P^2(\mathbf{R})$, one can calculate intersections of lines exactly as one calculates joining lines of points. We have not yet been formally introduced to *duality*, but we know enough to carry out some exercises in $P^2(\mathbf{R})$. To begin with we need to define a *dictionary of duality*, that is, a list of terms which correspond to each other. For $P^2(\mathbf{R})$, we call this *plane duality* and the dictionary looks like:

term	**dual term**
point (Punkt)	line (Gerade)
to lie in (liegen in)	to pass through (gehen durch)
join (verbinden)	intersect (schneiden)
collinear	coincident

Dualize (in the plane) the following, and make a sketch of both what is given and its dual:

- Two lines intersect in a unique point.
- Three points not lying in a given line, and their joining lines.
- Four lines and their six points of intersection.
- All the lines passing through a given point.
- Given two points lying on a line, all the points which can be reached from a third point on the line, without crossing either of the two given points.

The set referred to in the last example is called a *line segment*, and the dual is called a *fan* – *Winkel* auf deutsch.

Hausaufgaben

1. Aufgabe (5 Punkte)

Duality

i) Dualize the harmonic fourth construction from the first practice exercise on this sheet. Carry out the construction; the result should be a clearly labeled, accurate drawing, preferably with different colors used to mark different steps in the construction.

ii) Given a set of lines in general position in $P^2(\mathbf{R})$and a point P lying on none of the lines, the *kernel* containing P is the set of points Q in $P^2(\mathbf{R})$such that there exists a line segment joining P and Q with no point in common with any of the given lines. Dualize this construction for the case of three lines. The dual concept to *kernel* [D: Kern] is *hull* [D: Hülle]. You should find 4 hulls. Sketch each one separately.

2. Aufgabe (2 Punkte)

Joining and intersecting planes

i) Find the plane equation for the plane joining the points $P = (1, 1, 0, 1)$ and $Q = (0, -1, 1, 1), and (1, 0, 1, 1)$ in $P^3(\mathbf{R})$, using the determinant formula analogous to the case in $P^2(\mathbf{R})$.

ii) Find the point of intersection of the three planes $x + y = 1$ and $y - z = 2$, and $x + y + z + 1 = 0$ in $P^3(\mathbf{R})$, using the determinant formula analogous to the case of intersecting lines in $P^2(\mathbf{R})$.

3. Aufgabe (3 Punkte)

Incidence. This exercise deals with lines in $P^3(\mathbf{R})$.

i) Prove that two lines in general do not intersect in $P^3(\mathbf{R})$. Such lines are called *skew* lines.

ii) Given three lines which are pair-wise skew, prove that there are an infinite number of lines which intersect all three lines.

iii) Given two skew lines l and m, show that there exists exactly one plane containing l and one plane containing m such that l is parallel to m.

4. Aufgabe (6 Punkte)

Projective transformations

- If M is a matrix representing a projective transformation of $P^n(\mathbf{R})$, show that an eigenvector of M corresponds to a fixed point of the transformation in $P^n(\mathbf{R})$. Show that a projective transformation of $P^2(\mathbf{R})$has at least one fixed point.

- Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represent a projective transformation of $P^1(\mathbf{R})$.

  i) What are the eigenvalues of M. Under what conditions are they real and unequal? real and equal? conjugate imaginary? (These three cases as known as *hyperbolic*, *parabolic*, and *elliptic*, respectively).

  ii) Show that the three cases from the previous question are represented by the three matrices $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Describe and give sketches for the action of M in these three cases on $P^1(\mathbf{R})$.

  iii) You know that a projective transformation of $P^1(\mathbf{R})$is determined by the image of three points. Define three points $\mathbf{0} := (0,1), \mathbf{1} := (1,1), \infty := (1,0)$. Given three more points $R = (r,1)$, $Q = (q,1)$, and $S = (s,1)$, – all different – find a,b,c,d so that $M(\mathbf{0}) = Q, M(\mathbf{1}) = R, M(\infty) = S$.

Gesamtpunktzahl: 16