TECHNISCHE UNIVERSITÄT BERLIN WS2005
Fakultät II - Institut für Mathematik
Prof. Dr. U. Pinkall / Charles Gunn Abgabe: 2.12.2005

6. Übung Geometrie I

(Lines in $P^3(\mathbf{R})$)

Web site: http://www.math.tu-berlin.de/geometrie/Lehre/WS05/Geometrie1.
This assignment includes 28 points. so you can choose 12 points to ignore without incurring any penalty.

Hausaufgaben

1. Aufgabe (8 Punkte)

Plücker Coordinates Given two points $P := (x_1, x_2, x_3, x_4), Q = (y_1, y_2, y_3, y_4)$, define the *Plücker line coordinates* of the line $p = PQ$ to be $(p_{12}, p_{13}, p_{14}, p_{23}, p_{42}, p_{34})$ where $p_{ij} := x_i y_j - x_j y_i$. **Notice:** for p_{ij} we have $i < j$ except for p_{42}! The reason for this reversal will become clear below, hopefully.

It is also useful to dualize these coordinates. Suppose we are given two planes also containing the points P and Q above and hence the line l. Let them be $\pi := (u_1, u_2, u_3, u_4)$, $\mu = (v_1, v_2, v_3, v_4)$, define the Plücker *axis* coordinates of the line $q = \pi\mu$ to be $(\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{42}, \pi_{34})$ where $\pi_{ij} := u_i v_j - u_j v_i$.

Define the matrices $M(P, Q) := \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}$

and $\hat{M}(\pi, \mu) := \begin{pmatrix} 0 & \pi_{12} & \pi_{13} & \pi_{14} \\ -\pi_{12} & 0 & \pi_{23} & \pi_{24} \\ -\pi_{13} & -\pi_{23} & 0 & \pi_{34} \\ -\pi_{14} & -\pi_{24} & -\pi_{34} & 0 \end{pmatrix}$

As we saw in the lecture, we can consider p_{ij} as coordinates on $P(\wedge^2(V))$. Define an inner product on these coordinates by $\omega(p, q) := p_{12}q_{34} + p_{13}q_{42} + p_{14}q_{23} + p_{23}q_{14} + p_{42}q_{13} + p_{34}q_{12}$.

Use whatever techniques you find useful, to establish the following properties.

- When p represents the Plücker coordinates of a line, $\omega(p, p) = 0$. (This quadric in the 5-dimensional space $P(\wedge^2(V))$ is called the *Klein* quadric.)
- The i^{th} row/column of M represents the point coordinates of the intersection of the i^{th} coordinate plane with the line PQ (unless the line is contained in this coordinate plane). For example the first row is the intersection with the plane $x = 0$, etc.
- The i^{th} row/column of $\hat{M}$ represents the plane coordinates of the plane spanned by the i^{th} basis vector and the line $\pi\mu$ (unless the line contains the i^{th} basis vector).

- Show that the following relationship exists between *line* coordinates and *axis* coordinates of the same line:

$$(p_{12}, p_{13}, p_{14}, p_{23}, p_{42}, p_{34}) = \lambda(\pi_{34}, \pi_{42}, \pi_{23}, \pi_{14}, \pi_{13}, \pi_{12}), \lambda \neq 0$$

  That is, the axis coordinates taken in reverse order are proportional to the line coordinates. More to the point, $p_{ij} = \lambda \pi_{kl}$ where $(ijkl)$ is a permutation of (1234).

2. Aufgabe (2 Punkte)

- Show that ω, considered as a symmetric bilinear form on the 6-dimensional space $\wedge^2(V)$, has signature $(+++---)$.
- If $P = (x_0, y_0, z_0, 1)$ and $Q = (x_1, y_1, z_1, 1)$ are two points in $\mathbb{R}^3$, find the Plücker coordinates for the line PQ and interpret the coordinates geometrically (possibly after rearranging them into two vectors).
- **Optional**: For two lines in $\mathbb{R}^3$, calculate the distance between the two lines in terms of the Plücker coordinates of the two lines.

3. Aufgabe (4 Punkte)

Assume we have coordinates for V so that the matrices M and $\hat{M}$ are well-defined.

- Define a map $P : V \to V^*$ by $Pv(w) :=< \hat{M}^t v, w >$. (Recall that $\hat{M}$ was determined by the axis $q = \pi\mu$.) Show that $ker(Pv)$ is the plane spanned by q and v.
- Define a map $\hat{P} : V^* \to V$ by $P\rho := M^t \rho$. (Recall that M was determined by a line $p = PQ$.) Show that ρ is the intersection of p and the plane ρ.
- **Optional**: Calculate the intersection point of two lines p and q from the Plücker coordinates of p and q. [Hint: use the first two parts of this exercise.]

4. Aufgabe (6 Punkte)

- A plane pencil P_l [deutsch: Ebenenbuschel] is the set of all planes containing a given line l. l is the axis of the pencil. Given a plane pencil with axis l and a line m which doesn't intersect l, define a map $f : P_l \to m$ by $f(\pi) = \pi \cap m$, that is, the natural correspondence taking a plane to its intersection with m. Show that f is projective. [That means, show that when expressed in coordinates, f is given by a linear transformation]. f is called the perspectivity between P_l and m.

- You learned in class that a regulus [deutsch: Regelschar] consists of the joining lines of corresponding points in two lines related by a projectivity. Use the above to prove that the set of all lines meeting three skew lines in $P^3(\mathbf{R})$is a regulus.
- Let R be a regulus determined by two lines q and r. Let l and m be two lines belonging to the regulus. Show that there is a projectivity between l and m producing a regulus R' such that q and r belong to R' and every line belonging to R meets every line belonging to R'.

5. Aufgabe (4 Punkte)

- Prove that, in general, that at most two lines meet four given lines in $P^3(\mathbf{R})$.
- Calculate Plücker coordinates for all lines which meet the four lines given by the four point pairs

$$\begin{aligned}&((0,0,0,1),(1,1,0,1)),\\&((0,1,1,0),(1,0,1,0)),\\&((1,0,0,1),(0,0,1,1)),\\&((0,1,0,1),(1,1,1,1))\end{aligned}$$

Full credit can be given only for work based on Plücker line coordinates.

6. Aufgabe (4 Punkte)

- Let a line p have Plücker line coordinates p_{ij}. Assume the line lies in none of the coordinate planes $x_i = 0$. Let P_i be the point where p meets the coordinate plane $x_i = 0$. Show that $cr(P_0, P_1, P_2, P_3) = -\frac{p_{14}p_{23}}{p_{12}p_{34}}$.