TECHNISCHE UNIVERSITÄT BERLIN WS2005
Fachbereich 3 - Mathematik
Prof. Dr. U. Pinkall / Charles Gunn Abgabe: 16.12.2005

8. Übung Geometrie I

(Linear line complexes)

Web site: http://www.math.tu-berlin.de/geometrie/Lehre/WS05/Geometrie1.
Note: In this sheet we assume $dim(V) = 4$ unless otherwise noted. And there are 18 points, so 2 points can be ignored.

Übungsaufgaben

1. Aufgabe

Line complexes Review of line complexes. Two ways to define line complex. One, via hyperplanes in $P(\wedge^2(V))$. The other via null-systems in $P(V)$, that is, correlations (polarities) defined via skew-symmetric bilinear forms. Both start with an element $a \in P(\wedge^2(V))$. The one defines the polar hyperplane with respect to the Plücker quadric $H := \{b \in P(\wedge^2(V)) | \omega(a, b) = 0\}$. The linear complex is then defined $L_a := H \cap Q$. The second way defines a correlation $k_a : V \to V^*$ by $k(v)(w) := a \wedge v \wedge w(det)$. As in the case of quadrics, where the bilinear form is symmetric, the plane $k(v)$ is called the *polar* plane of v.

- Show that the two methods are equivalent.
- Given a coordinate basis for $P(V)$ give a matrix representation for k. Show that this matrix is non-singular $\iff$ $a \notin Q$.
- The inverse of k is a map $\hat(k) : V^* \to V$, which takes planes to their polar points.
- What's wrong with the following reasoning? "There are 3 dimensions of points in $P(V)$; through each point there is a 1 dimensional family of lines. Thus, a line complex consists of a 4-dimensional family of lines."

Hausaufgaben

1. Aufgabe (6 Punkte)

Polar lines Given a linear complex L_a defined as above. Suppose h is a line, not necessarily belonging to the complex. Last week's homework you should have shown that as a point moves along h, the polar plane rotates around a second line, called the *polar line* and denoted $\hat{h}$.

- Show that h and $\hat{h}$ either coincide (when $h \in L_a$), or they are skew [windschief].

- Show that when $h \notin L_a$, any line g that meets both h and $\hat{h}$ belongs to the complex, and if a line of the complex meets h, it also meets $\hat{h}$.
- Show that the lines polar to lines that pass through a point $v \in P(V)$ all lie in the polar plane $k(v)$ of v. And, dually, all the lines polar to lines lying in a plane $E \in P(V^*)$ pass through the polar point of E.
- Given $h \notin L_a$. Define a point $h' \in P(\wedge^2(V))$ by $h' :=< a, a > h - 2 < h, a > a$. Show that $h' \in Q$ and that every line of L_a that meets h also meets h'. Conclude that $h' = \hat{h}$, the polar line of h.

2. Aufgabe (6 Punkte)

Linear complex in Euclidean space This exercise is a continuation of Exercise 1 from last week. As there, we treat Euclidean space as coordinatized as $(1, x, y, z)$, that is, the homogenized coordinate is the first one. Then the Plücker coordinates for a line g can be represented by the two 3-vectors (X, Z) where $X = (p_{12}, p_{13}, p_{14}) = P_2 - P_1$ and $Z = (p_{34}, p_{42}, p_{23}) = P_1 \times P_2$. (Refer to the assignment if you need to). Then the Plücker inner product can be written $\omega(a, a) =< X, Z >= 0$. Let $a \in P(\wedge^2(V)) \setminus Q$, and L_a the associated linear complex. This exercise will explore this linear complex as it sits in $\mathbb{R}^3$.

- Let E_∞ be the ideal plane $w = 0$ (here, w is the first coordinate!), and $P \in E_\infty$ be its polar point with respect to the null system determined by a. Consider the line bundle of all lines passing through P. A line in this bundle is called a *diameter* of the complex. Show that the polar planes of the points lying on a given diameter are parallel to each other. Hint: the previous exercise.
- Show there is a unique diameter d such that all polar planes of points in d are perpendicular to d. This diameter is called the *axis* of the complex. Hint: use the previous exercise.
- Find the Plücker coordinates of the axis in terms of the coordinates (X, Z) of a.
- Apply your formula to find the axis of the linear complex given by $X = (1, 0, 1)$ and $Z = (0, 1, 1)$.
- What significance, if any, does the axis have when the linear complex is considered inside of $P^3(\mathbf{R})$ rather than Euclidean space?

3. Aufgabe (2 Punkte)

- Suppose an infinitesimal motion is defined on $\mathbb{R}^3$ by $Y(p) = a \times p + b$ where $a = (1, 1, 1)$ and $c = (2, 1, 3)$. Find m such that with new coordinates $\tilde{p} := p - m$ the vector field has the form $Y(\tilde{p}) = a \times \tilde{p} + \lambda a$ and calculate λ.

4. Aufgabe

(4 Punkte)

Linear congruences A linear line congruence is defined to be the intersection of a 3-plane in $P(\wedge^2(V))$ with the Plücker quadric Q. We saw in the lecture that such a 3-plane is determined by a 4-dimensional subspace $U \subset P(\wedge^2(V))$, so that $P(U^\perp)$ is a line $l \subset P(\wedge^2(V))$. l can intersect Q in 2, 1, or 0 points.

- Suppose the intersection consists of two distinct points a and b.
  - Verify that the congruence consists of all lines which intersect both a and b.
  - Points on l can be written $a + \lambda b$ for $\lambda \in \hat{\mathbb{R}}$. For each point, we can consider its polar hyperplane and the associated linear complex. Show that each of these linear complexes contains all the lines of the congruence.
- Describe the linear congruence in case the intersection consists of 1 point.

Gesamtpunktzahl: 18