# 1 Hyperbolic isometries

An isometry of hyperbolic n-space is an element of $O(n,1)$. This is the group of matrices which preserve the quadratic form $(+++...++-)$ which $n$ +'s and 1 $-$. To make life simpler, we work out our examples in $n=2$, the hyperbolic plane, but do as much of the theory as possible in the general n-dimensional setting. For the hyperbolic plane the quadratic form in matrix form is $Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.
Let the inner product with respect to $Q$ be written as $\langle .,. \rangle_Q$.
**Achtung:** Note that, as usual, I am using the *last* coordinate as the distinguished coordinate (the one corresponding to the negative sign), while in the lecture the *first* coordinate served this purpose.
Then

$$A \in O(n,1) \iff \langle Av, Aw \rangle_Q = \langle v, w \rangle_Q \ \forall \ (v,w) \in V.$$

In matrix form this equation looks like:

$$v^t A^t Q A w = v^t Q w \ \forall \ (v,w) \iff A^t Q A = Q$$

The last equation gives a useful formula for the inverse of $A$: $A^t Q A = Q \Rightarrow A^{-1} = Q A^t Q$. (Here, we use the fact that $Q^{-1} = Q$.) We use this formula below in the last example.
Letting $v = e_i$ and $w = e_j$ and defining $A_i$ be the $i^{th}$ column of $A$, considered as a vector. Then

$$\langle e_i, e_j \rangle_Q = \langle A_i, A_j \rangle_Q = \left\{ \begin{array}{ll} -1 & \text{if } (i,j) = (n+1, n+1) \\ \delta_{ij} & \text{otherwise} \end{array} \right\} \tag{1}$$

In other words, the columns of $A$ form an orthonormal basis of $H^n$ with respect to the inner product $Q$, and any orthonormal basis with respect to $Q$ can be used to construct such $A \in O(2,1)$. Note that in this orthonormal basis, all vectors satisfy $\langle u_i, u_i \rangle_Q = 1$ except $\langle u_{n+1}, u_{n+1} \rangle_Q = -1$. That is, only the last vector is an element of the hyperbolic space itself.
**Example:** Show that the identity matrix $\mathbb{1} \in O(2,1)$.

> Calculate the inner products of the columns with each other, with respect to $\langle .,. \rangle_Q$, and verify these inner products satisfy the above conditions.

**Example:** Calculate $A \in O(2,1)$ which maps the standard basis to the orthonormal basis $(u_1, u_2, u_3)$ of $H^2$ such that $u_3$ is equivalent to $v = (0,1,2)$ and such that $u_2$ and $u_3$ span the line through $v$ and $w = (1,0,2)$.

> First, normalize $(0,1,2)$ to get $u_3 = \frac{1}{\sqrt{3}}(0,1,2)$. To get $u_2$, one has to apply the Gram-Schmidt process with the hyperbolic inner product.

That is, one seeks a vector $x := v + tw$ such that $\langle x, v\rangle_Q = 0$. Once $x$ is found, one can normalize it.) To find $x$, calculate:

$$\begin{aligned} 0 = \langle x, v\rangle_Q &= \langle v, v\rangle_Q + t\langle w, v\rangle_Q \\ &= -3 + t(-4) \Rightarrow \\ t &= -\frac{4}{3} \Rightarrow \\ x &= (1, -\frac{4}{3}, -\frac{2}{3}) \end{aligned}$$

One verifies easily that $\langle x, v\rangle_Q = 0$. $\langle x, x\rangle_Q = \frac{21}{9}$ means that the normalized form is $u_2 = \frac{1}{\sqrt{21}}(3, -4, -2)$. To finish the basis we need to find $u_1$ perpendicular (hyperbolically!) to both $u_2$ and $u_3$. In euclidean geometry, we could calculate the cross product $u_2 \times u_3 = k(6, 6, -3)$. For hyperbolic geometry, we only need to flip the last coordinate to get the desired behavior. It is easy to verify that $(6, 6, 3)$ is perpendicular to both $u_2$ and $u_3$. Then the normalized version is $u_1 = \frac{1}{\sqrt{7}}(2, 2, 1)$. The desired $A$ is just the matrix with this basis as its columns:

$$A := \begin{pmatrix} \frac{2}{\sqrt{7}} & \frac{3}{\sqrt{21}} & 0 \\ \frac{2}{\sqrt{7}} & -\frac{4}{\sqrt{21}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{7}} & -\frac{2}{\sqrt{21}} & \frac{2}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{21}} \begin{pmatrix} 2\sqrt{3} & 3 & 0 \\ 2\sqrt{3} & -4 & \sqrt{7} \\ \sqrt{3} & -2 & 2\sqrt{7} \end{pmatrix}$$

**Example:**(Similar to Assg 9, Ex. 3) Calculate the element $A \in O(2, 1)$ which takes the point $P_1 := (0, 0, 1)$ and the line $g$ through $P_1$ and $Q_1 := (1, 1, 2)$ to the point $P_2 := (-1, 0, 2)$ and the line $h$ through $P_2$ and $Q_2 := (0, -1, 2)$.

Construct matrix $B$ as above which maps the standard basis to the orthonormal basis determined by $P_1$ and $Q_1$, and similarly construct $C$ which maps the standard basis to the orthonormal basis determined by $P_2$ and $Q_2$. Then the desired matrix $A$ is the product $A := CB^{-1}$. To calculate $B^{-1}$, use the formula derived above.