TECHNISCHE UNIVERSITÄT BERLIN
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Complex Analysis II / Geometry III - WS 12
Riemann Surfaces: Classical and Discrete

www3.math.tu-berlin.de/geometrie/Lehre/WS12/ComplexAnalysisII/

Exercise Sheet 12

Exercise 1. (6 points)

Let $\mathcal{R}$ be a discrete compact Riemann surface of genus $g > 0$ obtained from a simplicial decomposition Γ and weights $\nu \in \mathbb{R}$. Let $f = u + iv$ be a discrete Abelian integral of the first kind with periods $A_1, B_1, \ldots, A_g, B_g$. Show for the Dirichlet energy of u that

$$E(u) = \frac{1}{2}\langle u, v\rangle = -\frac{1}{2}\operatorname{Im}\left(\sum_{k=1}^{g} A_k \overline{B_k}\right).$$

Exercise 2. (7 points)

a) Show that there exists a triangulation $\mathcal{T}$ of a compact Riemann surface of genus g which cannot be cut to a triangulation of the simply connected model F_g.

b) Prove Riemann's bilinear identity for any discrete compact Riemann surface $\mathcal{R}$ of genus $g > 0$ obtained from a simplicial decomposition Γ and real weights ν.

Hint: For the first part, choose a triangulation with maximum vertex degree less than $4g$ for sufficiently high genus g. In the second part, you can use (without proof) that for any two triangulations $\mathcal{T}, \mathcal{T}'$ of a compact Riemann surface of genus g, there exists a common refinement, i.e. a triangulation $\mathcal{T}''$ such that the triangles of $\mathcal{T}, \mathcal{T}'$ are triangulated by the triangles of $\mathcal{T}''$.

Exercise 3. (7 points)

Let $\mathcal{R}$ be a discrete compact Riemann surface with complex weights ν having positive real part. Denote by D the underlying quad-graph with black vertices $V(\Gamma)$ and white vertices $V(\Gamma^*)$. For $H : V(D) \to \mathbb{R}$ we define the *discrete Dirichlet energy* as

$$E(H) := \sum_{e \in E(\Gamma)} \frac{1}{4\operatorname{Re}(\nu(e))}\{|\nu(e)|^2 (H(h_e) - H(t_e))^2 + (H(l_e) - H(r_e))^2 + 2\operatorname{Im}(\nu(e))(H(h_e) - H(t_e))(H(l_e) - H(r_e))\}.$$

a) Show that E is a strictly convex non-negative functional quadratic in H if one fixes the value of H on each a vertex of Γ and Γ^*.

b) Prove $-\partial E(H)/\partial H(x) = \triangle H(x)$ where $\triangle$ is the discrete Laplacian defined in Exercise Sheet 11, Exercise 3.

c) Deduce Liouville's theorem: Any discrete holomorphic function on $\mathcal{R}$ is essentially constant, i.e. constant on $V(\Gamma)$ and constant on $V(\Gamma^*)$.

Due: Tutorial on 6 February, 2013.