TECHNISCHE UNIVERSITÄT BERLIN
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Complex Analysis II / Geometry III - WS 12
Riemann Surfaces: Classical and Discrete

www3.math.tu-berlin.de/geometrie/Lehre/WS12/ComplexAnalysisII/

Exercise Sheet 13

Exercise 1. (6 points)

Let $\mathcal{R}$ be a (smooth) compact Riemann surface of genus g, let $a_1, b_1, \dots, a_g, b_g$ be a canonical homology basis, and $\omega_1, \dots, \omega_g$ a dual basis of holomorphic differentials. Show that for any real numbers $A_1, B_1, \dots, A_g, B_g$, there exists a unique holomorphic differential ω such that

$$\operatorname{Re}\left(\int_{a_k} \omega\right) = A_k \text{ and } \operatorname{Re}\left(\int_{b_k} \omega\right) = B_k \quad \forall k = 1, \dots, g.$$

Exercise 2. (7 points)

Let $\mathcal{T}$ be the triangulation obtained from the side surface of a regular square pyramid (i.e. all side lengths are equal) by identifying the opposite sides of the basis, and denote by $\mathcal{R}$ the corresponding discrete compact Riemann surface of genus one. Show that the discrete period matrices of $\mathcal{T}$ are given by

$$\Pi_{\mathcal{T}} = \frac{2}{\sqrt{3}} i \quad \text{and} \quad \Pi_{\mathcal{T}^*} = \frac{\sqrt{3}}{2} i.$$

Exercise 3. (7 points)

Let $\mathcal{R}$ be a discrete Riemann surface with complex weights ν having positive real part. Denote by D the underlying quad-graph. Let $D_0 \subset D$ be finite and homeomorphic to a disk. Denote by $V(D_0)$ the set of vertices, and by $V(\partial D_0)$ the set of boundary vertices.

For an interior vertex $x_0 \in V(D_0) \backslash V(\partial D_0)$, a real function $G_{D_0}(\cdot; x_0)$ on $V(D_0)$ is a *discrete Green's function in* D_0 if and only if

$$G_{D_0}(x; x_0) = 0 \quad \forall x \in V(\partial D_0) \text{ and } \triangle G_{D_0}(x; x_0) = \delta_{x x_0} \quad \forall x \in V(D_0) \backslash V(\partial D_0).$$

Here $\triangle$ is the discrete Laplacian defined in Exercise Sheet 11, Exercise 3.

Show that for any $x_0 \in V(D_0) \backslash V(\partial D_0)$ there exists a unique discrete Green's function $G_{D_0}(\cdot; x_0)$ in D_0.

Hint: Consider $\triangle$ as a linear operator $\mathbb{R}^{V(D_0)} \to \mathbb{R}^{V(D_0) \backslash V(\partial D_0)}$ and prove that it is surjective. You can use that the discrete Dirichlet boundary value problem on D_0 (i.e. looking for a discrete harmonic function on $V(D_0)$ having prescribed real values on $V(\partial D_0)$) is uniquely solvable – this follows from Exercise Sheet 12, Exercise 3, and does not have to be proven another time.

Due: Tutorial on 13 February, 2013.