TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. Alexander Bobenko
Felix Günther

Complex Analysis II / Geometry III - WS 12
Riemann Surfaces: Classical and Discrete

www3.math.tu-berlin.de/geometrie/Lehre/WS12/ComplexAnalysisII/

Exercise Sheet 5

Exercise 1. (5 points)

Finish the proof of Riemann's bilinear identity, i.e., show that $\sum_i (A_i B_i' - A_i' B_i)$ with $A_i = \int_{a_i} \omega$, $A_i' = \int_{a_i} \omega'$, $B_i = \int_{b_i} \omega$, $B_i' = \int_{b_i} \omega'$ for closed forms ω, ω' is independent of the choice of the canonical homology basis $a_1, b_1, \ldots, a_g, b_g$ on a compact Riemann surface of genus g.

Exercise 2. (5 points)

Deduce from Liouville's theorem that there exists no non-trivial holomorphic one-form on the Riemann sphere $\hat{\mathbb{C}}$.

Exercise 3. (5 points)

Let $\mathcal{R}$ be a compact Riemann surface and denote by $H^1_{dR}(\mathcal{R}, \mathbb{C})$ the factor space of closed one-forms modulo exact one-forms.

a) Prove that the canonical map $H^1(\mathcal{R}, \mathbb{C}) \to H^1_{dR}(\mathcal{R}, \mathbb{C})$ is well-defined and injective.

b) Let ω be a closed one-form and $\omega = p dz + q d\bar{z}$ in the local coordinate z. Show that $\omega - df$ for a function f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = q$ is satisfied.

Exercise 4. (5 points)

Consider an octagon $G_2 \subset \mathbb{C}$ with vertices on the eighth roots of unity where opposite edges are identified by the corresponding translations. We equip the corresponding compact Riemann surface $\mathcal{R}$ of genus 2 with the complex structure inherited from $\mathbb{C}$.

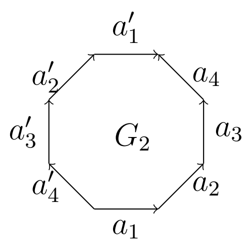

Figure 1: Octagon G_2

a) Show that dz induces a holomorphic one-form ω on $\mathcal{R}$. Calculate its periods with respect to a canonical homology basis.

b) Compute $\int_{\mathcal{R}} \omega \wedge \overline{\omega}$ using Riemann's bilinear identity and compare with the result you get from a direct computation.

Due: Tutorial on 28 November, 2012.