TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. Alexander Bobenko
Felix Günther

Complex Analysis II / Geometry III - WS 12
Riemann Surfaces: Classical and Discrete

www3.math.tu-berlin.de/geometrie/Lehre/WS12/ComplexAnalysisII/

Exercise Sheet 6

Exercise 1. (6 points)

Consider a regular hexagon $H \subset \mathbb{C}$ with vertices on the sixth roots of unity where opposite edges are identified by the corresponding translations. Compute the period matrix of the compact Riemann surface corresponding to H with the complex structure inherited from $\mathbb{C}$.

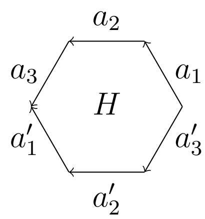

Figure 1: Regular hexagon H

Exercise 2. (7 points)

Let $\omega_1, \omega_2, \omega_1', \omega_2' \in \mathbb{C}$ such that $\frac{\omega_2}{\omega_1}, \frac{\omega_2'}{\omega_1'}$ lie in the upper half plane. We consider the lattices $\Lambda := \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}, \Lambda' := \{m\omega_1' + n\omega_2' | m, n \in \mathbb{Z}\}$ and want to determine when the complex tori $T := \mathbb{C}/\Lambda$ and $T' := \mathbb{C}/\Lambda'$ are biholomorphic.

a) Show that $\text{Sp}(1, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})$.

b) Let $\omega_1 = \omega_1' = 1$. Compute the periods of T and T' and conclude that the tori are not biholomorphic if there is no $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ such that $\omega_2' = \frac{a\omega_2 + b}{c\omega_2 + d}$.

c) Show that Λ and Λ' define the same lattice in $\mathbb{C}$ iff there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ such that $\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$.

d) Conclude that T and T' are biholomorphic iff there is $\mu \in \mathbb{C}$ such that $\Lambda' = \mu\Lambda$.

Exercise 3. (7 points)

Let $\mathcal{R}$ be the compact Riemann surface associated to a hyperelliptic curve of even degree $N = 2g+2$ with branch points $\lambda_1, \lambda_2, \ldots, \lambda_{2g+2}$. Find normalized Abelian differentials $\Omega^{(2)}_{\lambda_i}$, $i = 1, \ldots, 2g+2$, and $\Omega_{\lambda_{2j+1}, \lambda_{2j+2}}$, $j = 0, \ldots, g$, with respect to a canonical homology basis of your choice.

Due: Tutorial on 5 December, 2012.