TECHNISCHE UNIVERSITÄT BERLIN
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Complex Analysis II / Geometry III - WS 12
Riemann Surfaces: Classical and Discrete

www3.math.tu-berlin.de/geometrie/Lehre/WS12/ComplexAnalysisII/

Exercise Sheet 8

Exercise 1. (6 points)

Let P_1, P_2 be two not necessarily distinct points on the Riemann sphere $\hat{\mathbb{C}}$. Find all Abelian differentials Ω on $\hat{\mathbb{C}}$ with divisor $(\Omega) = mP_1 + nP_2$ for some $m, n \in \mathbb{Z}$.

Exercise 2. (6 points)

Let $\mathcal{R}$ be a compact Riemann surface of genus g and $D = D_0 - D_\infty$ a divisor splitted into positive and negative parts. Show that the dimension of the complex vector space $L(D)$ is bounded by $\deg(D_\infty) + 1$, and that the dimension of $H(D)$ is bounded by $g + \deg(D_\infty) - 1$ if $D_\infty \neq 0$ and by g if $D_\infty = 0$. In particular, both spaces are finite-dimensional.

Exercise 3. (8 points)

Let $\mathcal{R}$ be a compact Riemann surface of genus $g > 0$ and let $x \in \mathcal{R}$. For an integer $n \geq 0$ we define the divisor $D_n := nx$.

a) Show that $l(-D_1) = l(-D_0)$.

b) Prove for any $n > 0$ that the difference $l(-D_n) - l(-D_{n-1})$ is either 0 or 1.

c) Show that the number of positive integers $n \leq N$ fulfilling $l(-D_n) - l(-D_{n-1}) = 1$ equals $N + i(D_N) - g$.

d) Prove that $i(D_N) = 0$ if $N > 2g - 2$.

e) Conclude that there are exactly g integers $n_1 < n_2 < \ldots < n_g$ such that there exist no meromorphic function on $\mathcal{R}$ having one pole of order n_k at x and no further poles. Moreover, $n_1 = 1$ and $n_g < 2g$.

Due: Tutorial on 9 January, 2013.