Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 2

(Tangent bundle)

due 31.10.2012

Exercise 1 4 points

Let $M$ be a topological manifold. Show that $M$ has an exhaustion by compact sets, i.e. a sequence $K_1 \subset K_2 \subset K_3 \subset \cdots \subset M$ of compact sets with $K_1 \subset K_{i+1}$ and $\bigcup_{i=1}^{\infty} K_i = M$.

Exercise 2 4 points

Let $M$ and $N$ be smooth manifolds of dimension $m$ and $n$ respectively. The topological product $M \times N$ is a topological manifold of dimension $m+n$. Let $\{(U_i, \varphi_i)\}_{i \in I}$ a smooth atlas for $M$ and $\{(V_j, \psi_j)\}_{j \in J}$ a smooth atlas for $N$. Show that $\{(U_i \times V_j, \varphi_i \times \psi_j)\}_{(i,j) \in I \times J}$ is a smooth atlas for $M \times N$. Here $\varphi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n$ with $(\varphi_i \times \psi_j)(x, y) = (\varphi_i(x), \psi_j(y))$.

Exercise 3 7 points

Let $M$ be a smooth manifold of dimension $n$. For each chart $\varphi : U \to \mathbb{R}^n$ we define $TU := \bigcup_{p \in U} T_p M \subset TM$ and a map $\Phi : TU \to \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ by $T_p M \ni X_p \mapsto (\varphi(p), X_p \varphi) \in \mathbb{R}^{2n}$.

Show:

i) $\Phi$ is bijective.

ii) If $\Psi : TV \to \psi(V) \times \mathbb{R}^n$ is another such map corresponding to a chart $\psi : V \to \mathbb{R}^n$ and $W := U \cap V \neq \emptyset$, then $\Psi \circ \Phi^{-1} : \Phi(TW) \to \Psi(TW)$ is a diffeomorphism.

Remark: With the induced topology, i.e. $A \subset TM$ is open if $\Phi(A \cap TU) \subset \mathbb{R}^{2n}$ is open for each such chart $\Phi : TU \to \varphi(U) \times \mathbb{R}^n$, $TM$ becomes a smooth manifold of dimension $m+n$ – the tangent bundle.